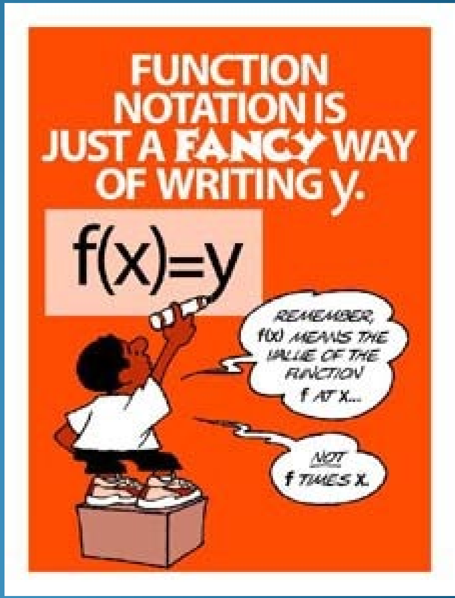
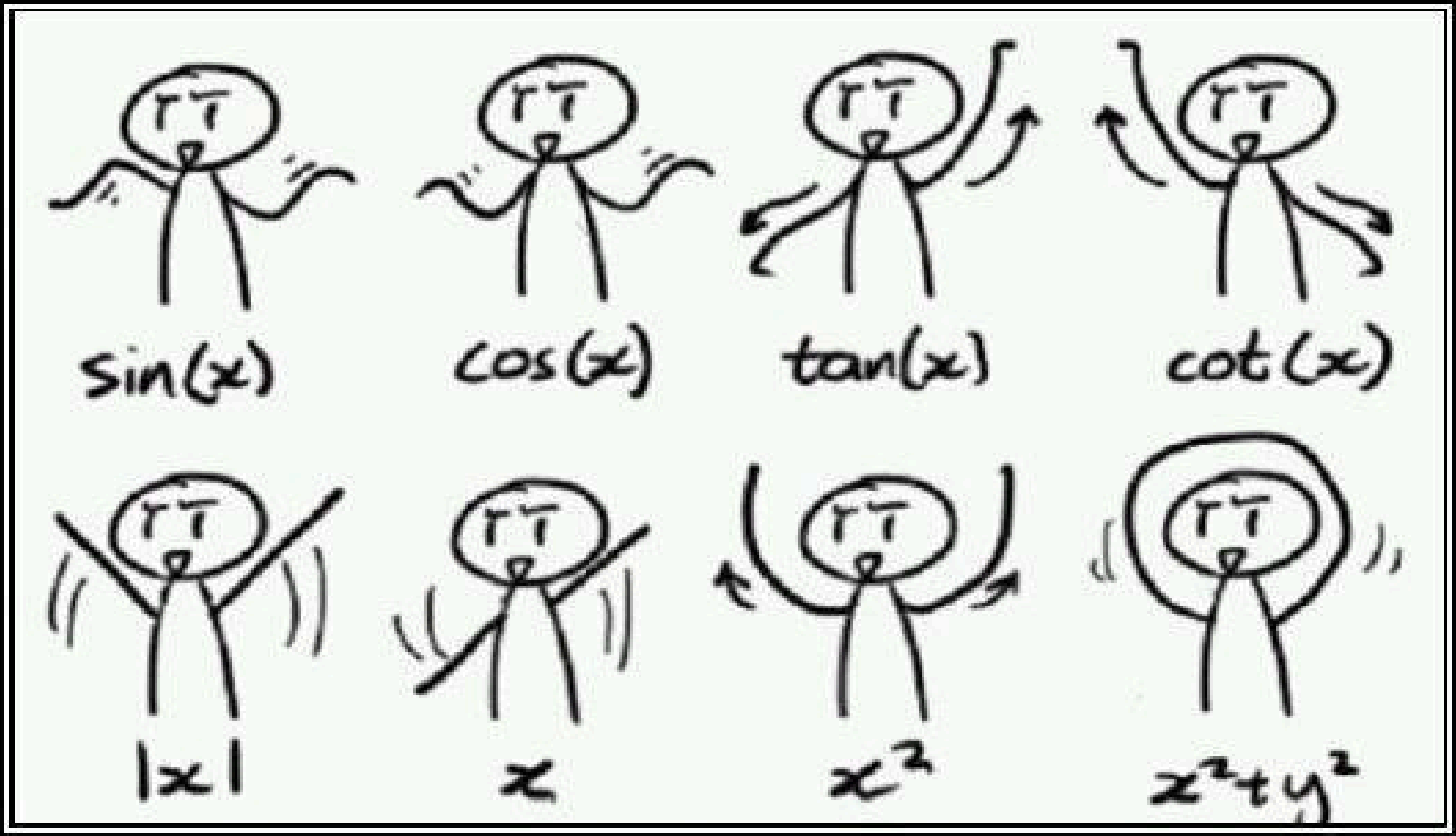
Lecture 1

FUNCTIONS





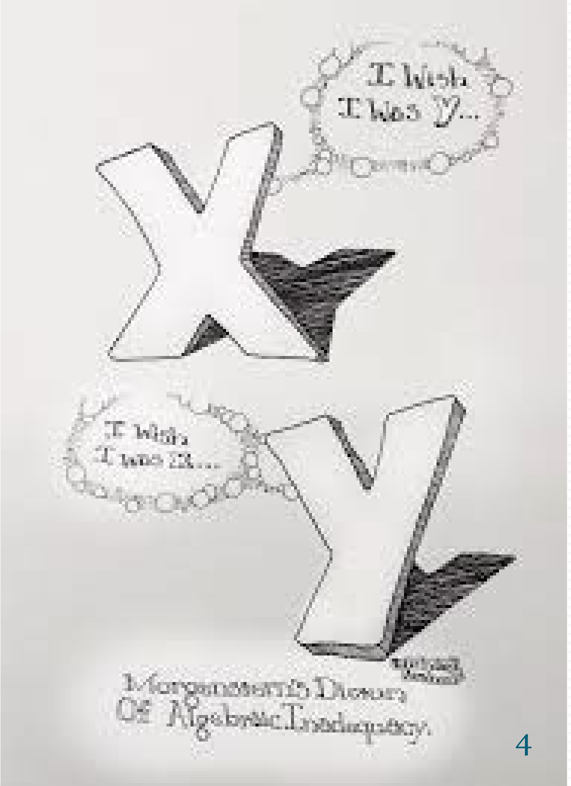
Dance lessons

for mathematicians

Subtopics

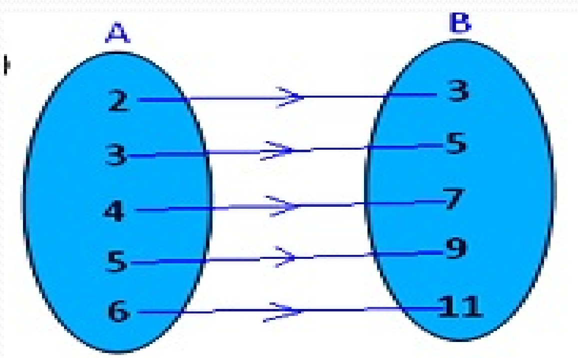
1. **Relations and Functions**
2. **Representation of Functions**
3. **New Function form Old Function**
4. **Inverse of Functions**
5. **Exponential Functions** e^x
6. **Logarithm Functions, log *x***

1.Relations and Functions
2.Representation of Functions



Relations and Functions

- **Definition-**A **function** - a **relation** in which map every element in the **domain** to an image in the **range**.
- A function is 1-1 relation and many-1 relation



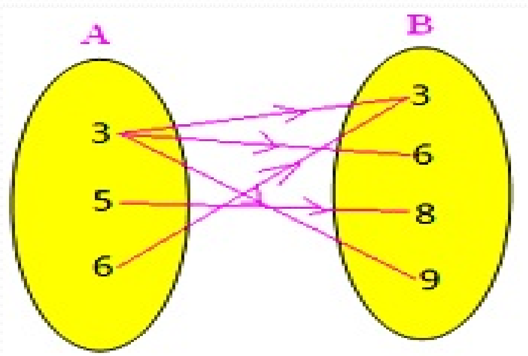


Image Range Image Range

Representation of Functions

1. Verbally (by a description in words)

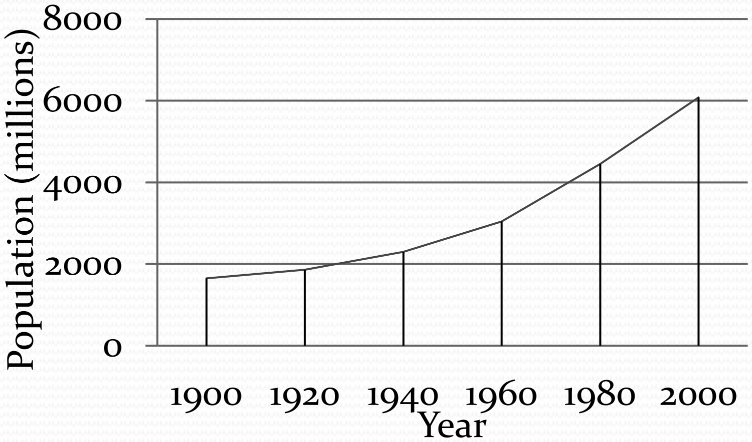
P(t) is the human population of the world of time

2. Numerically (by a table of values)

Year	1900	1920	1940	1960	1980	2000
Population (millions)	1650	1860	2300	3040	4450	6080

Representation of Functions

3. Visually (by a **graph**)



4. Algebraically (by an **explicit formula**)

$$P(t) \approx f(t) = (0.008079266) \cdot (1.013731)^t$$

Example 1:

- Let A = {1, 2, 3, 4} and B = {set of integers}. Illustrate the function $f: x \to x+3$.

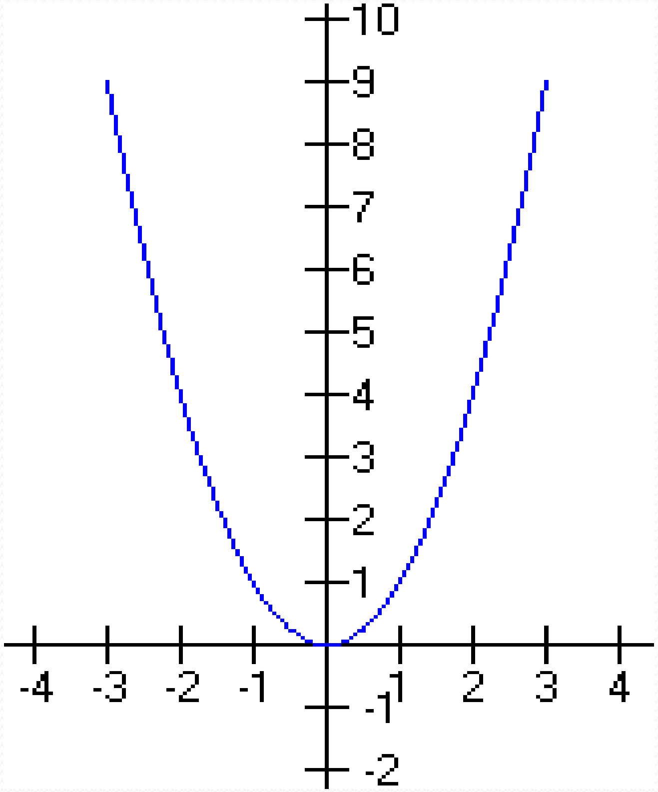
Example 2:

- Draw the graph of the function , $f : x \to x^2, x \in R$ where R is the set of real numbers.

Solution
Assume the domain is x = -3, -2, -1, 0, 1, 2, 3.
A table of values is constructed as follows:

x	-3	-2	-1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9

Example 2: **Graph**



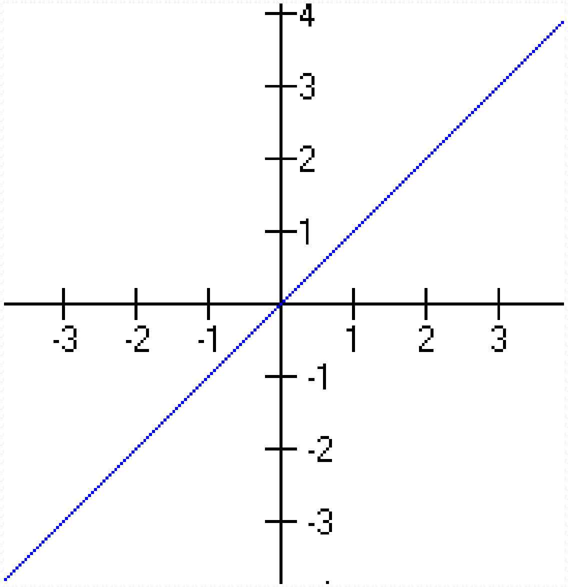
Type of Function and Their Graph

Linear Function

$$f(x) = a_1 x + a_0$$

$$f(x) = x \qquad ; x \in R$$

- Where a_1 and a_0 are constant called the coefficients of the linear equation



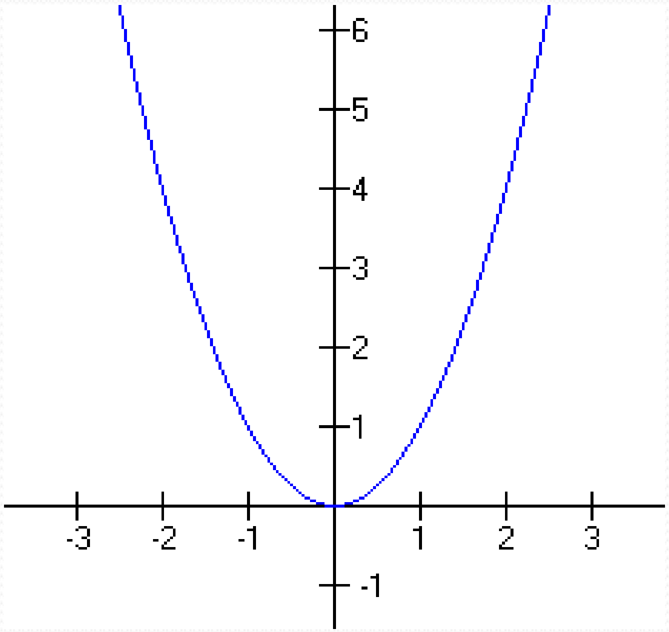
Type of Function and Their Graph

Polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x^1 + a_0$$

- Where n is a nonnegative integer and the number are constant $a_n, a_{n-1}, \cdots, a_2, a_1, a_0$ called the coefficients of the polynomial.

- <u>Quadratic</u> $f(x) = x^2 ; x \in R$



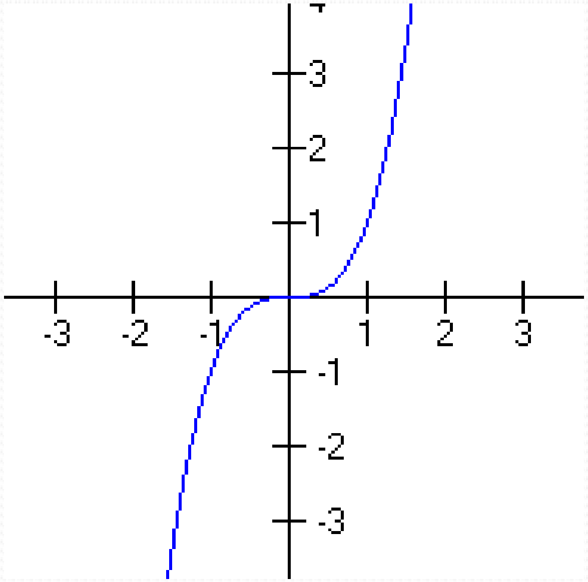
Type of Function and Their Graph

Power Function

$$f(x) = x^{a}$$

Where *a* is constant.

$$f(x) = x^3; x \in R$$



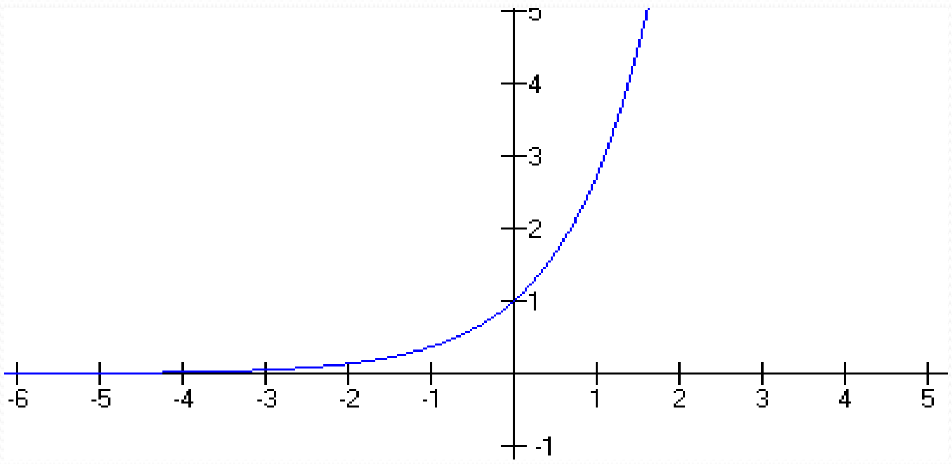
Type of Function and Their Graph

Exponential Function

$$f(x) = a^x$$

$$f(x) = e^x ; x \in R$$

Where a is a positive constant.



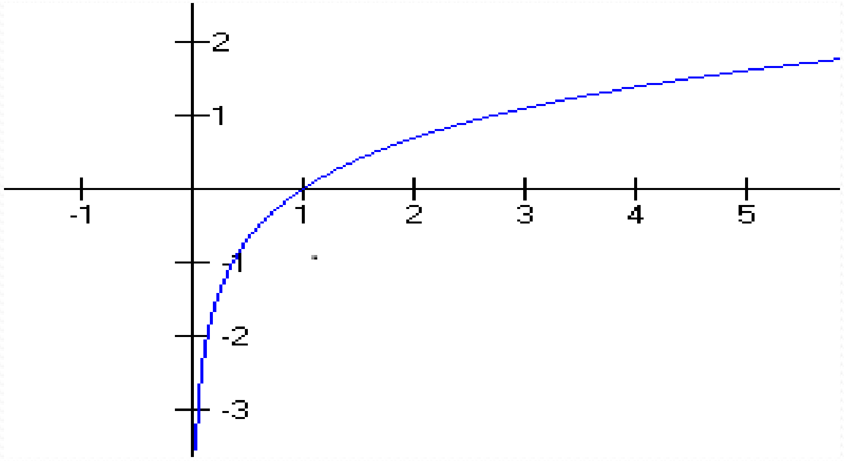
Type of Function and Their Graph

Logarithm Function

$$f(x) = \log_a x$$

Where a is a positive constant.

$$f(x) = \ln x; x \in (0, \infty)$$



Example 10:

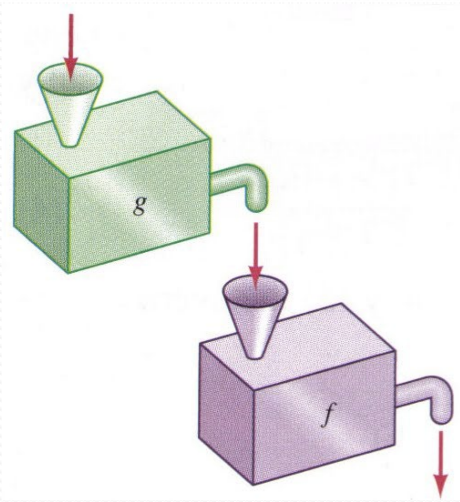
Consider for what value of x are the following function defined?

$$f(x) = 2x - 5$$

$$f(x) = \frac{1}{x-2}$$

3. New Functions from Old Function

1. **TRANSFORMATIONS OF FUNCTIONS**
2. **COMBINATION OF FUNCTIONS**
3. **COMPOSITE FUNCTIONS**



New Functions from Old Function

- **TRANSFORMATIONS OF FUNCTIONS**
- The graph of one function can be transform into the graph of a different function rely on a function's equation.

Vertical and horizontal shift

Let f be a function and c a positive real number.

$y = f(x) + c$ is the graph of $y = f(x)$ shifted c units vertically upward.

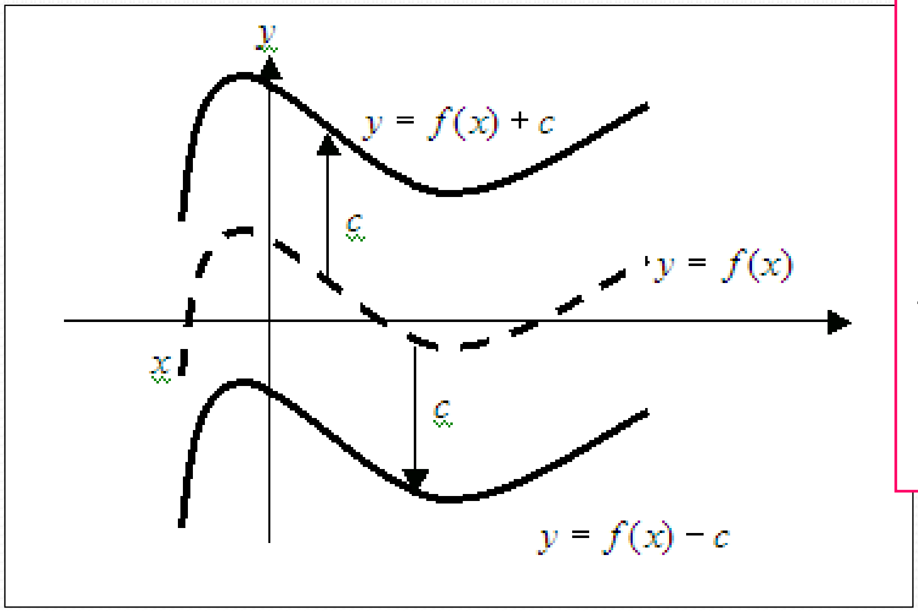
$y = f(x) - c$ is the graph of $y = f(x)$ shifted c units vertically downward.

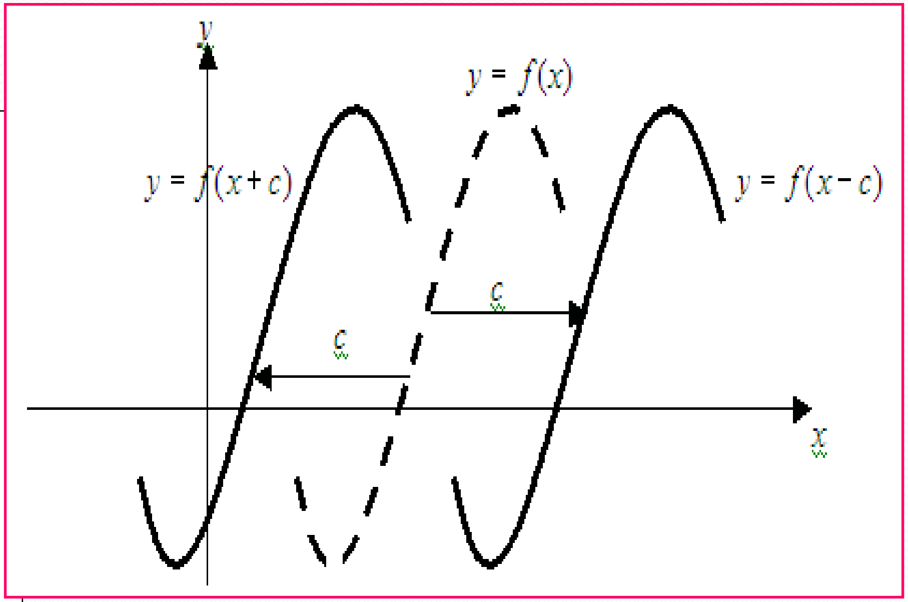
$y = f(x + c)$ is the graph of $y = f(x)$ shifted to the left c units.

$y = f(x - c)$ is the graph of $y = f(x)$ shifted to the right c units.

TRANSFORMATIONS OF FUNCTIONS

- Vertical and horizontal shift





Example 3:

Use the graph of $f(x) = |x|$ to obtain the graph of

$$g(x) = |x| - 4$$

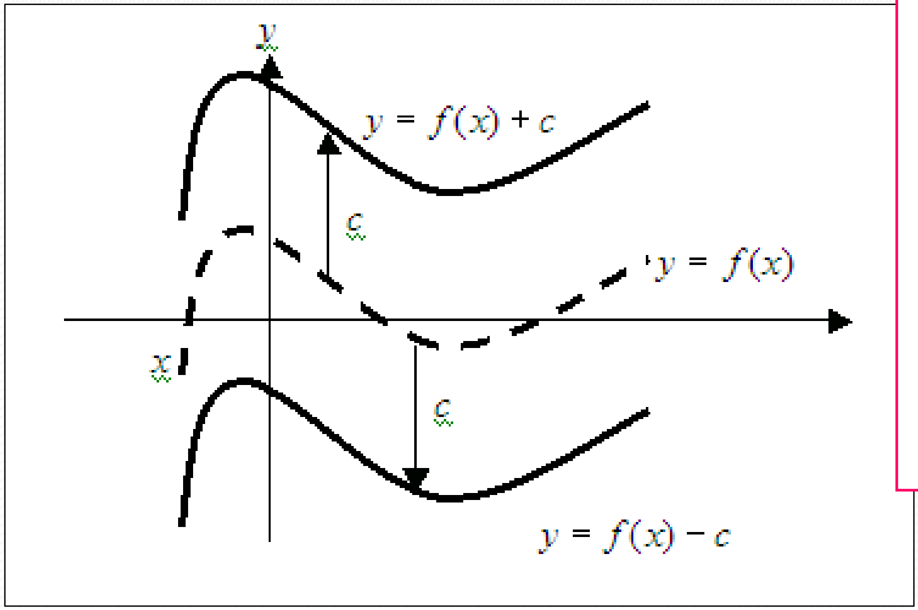
Example 4:

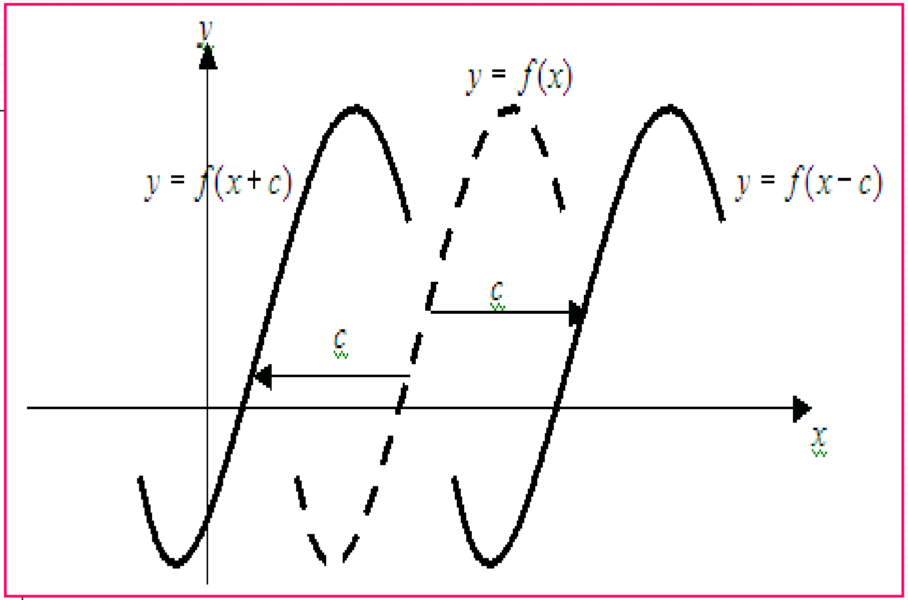
Use the graph of $f(x) = x^2$ to obtain the graph of

$$g(x) = (x+2)^2$$

TRANSFORMATIONS OF FUNCTIONS

- Vertical and horizontal shift





TRANSFORMATIONS OF FUNCTIONS

Vertical and Horizontal Reflecting and Stretching

Let f be a function and $c > 1$.

$y = -f(x)$	is the graph of $y = f(x)$ reflected about x-axis.
$y = f(-x)$	is the graph of $y = f(x)$ reflected about y-axis.
$y = cf(x)$	is the graph of $y = f(x)$ vertically stretched by multiplying each of its y-coordinates by c. if $0 < c < 1$, is the graph of $y = f(x)$ vertically shrunk by multiplying each of its y-coordinates by c.
$y = f(\frac{x}{c})$	is the graph of $y = f(x)$ horizontally stretched by multiplying each of its y-coordinates by c.
$y = \left(\frac{1}{c}\right)f(x)$	is the graph of $y = f(x)$ vertically compressed by multiplying each of its y-coordinates by c.
$y = f(cx)$	is the graph of $y = f(x)$ horizontally compressed by multiplying each of its y-coordinates by c.

Example 5:

Use the graph of $f(x)=\sqrt{x}$ to obtain the graph of

$$g(x)=-\sqrt{x}$$

$$h(x)=\sqrt{-x}$$

Example 5:

Use the graph of $f(x) = x^2$ to obtain the graph of

$$g(x) = 2x^2$$

$$h(x) = \frac{1}{2}x^2$$

COMBINATION OF FUNCTIONS – The algebra of functions

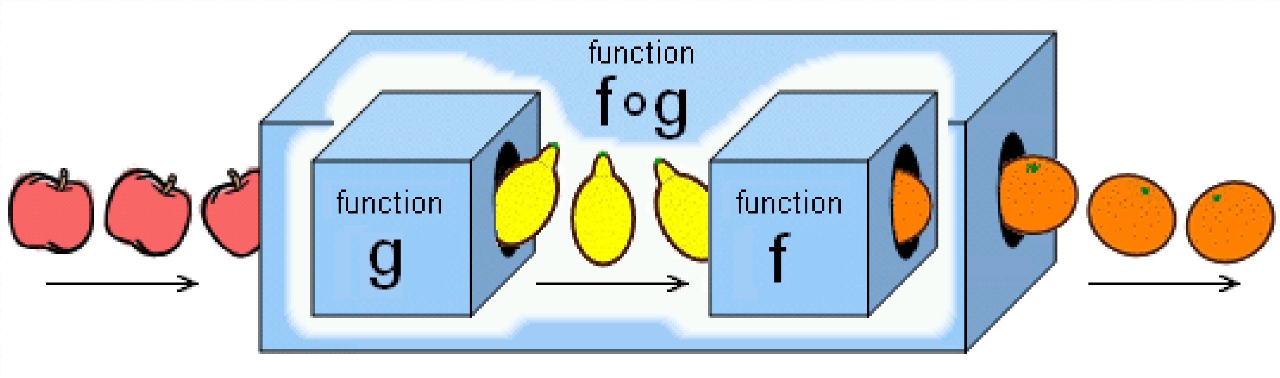
- Functions can be added, subtracted, multiplied and divided in a many ways.

For example consider $f(x) = x^2$ and $g(x) = 2x + 6$.

a) f(x)+g(x)	and	g(x)+f(x)
b) f(x)-g(x)	and	g(x)-f(x)
c) f(x)/g(x)	and	g(x)/f(x)
d) f(x).g(x)	and	g(x).f(x)

COMPOSITE FUNCTIONS

- **Definition-** Consider two functions $f(x)$ and $g(x)$. We define $f \circ g = fg(x) = f[g(x)]$ meaning that the output values of the function *g* are used as the input values for the function *f*.



Example 6:

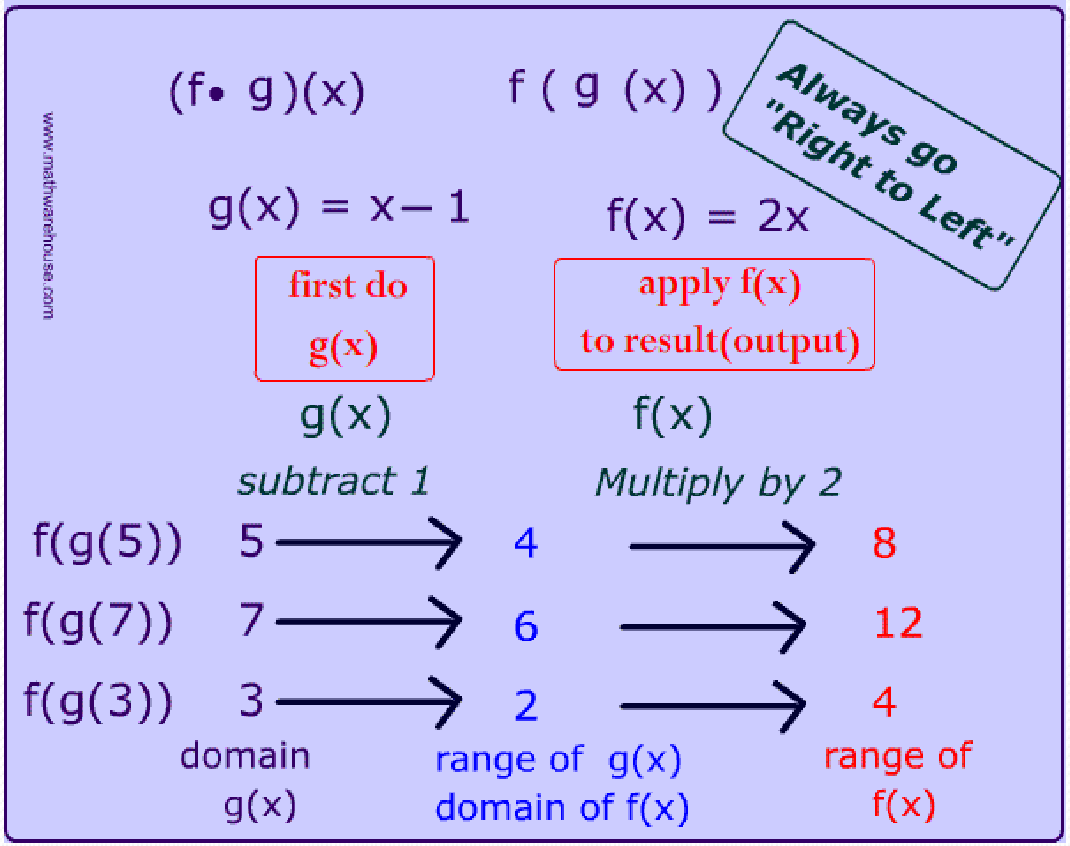
- If $f(x) = 3x + 1$ and $g(x) = 2 - x$, find as a function of x

(a) $f \circ g$

(b) $g \circ f$

COMPOSITE FUNCTIONS

- **Determine the Domain of the Composite Functions**



Example 7:

- If $f(x)=3x+1$ and $g(x)=2-x$, find as a function of x

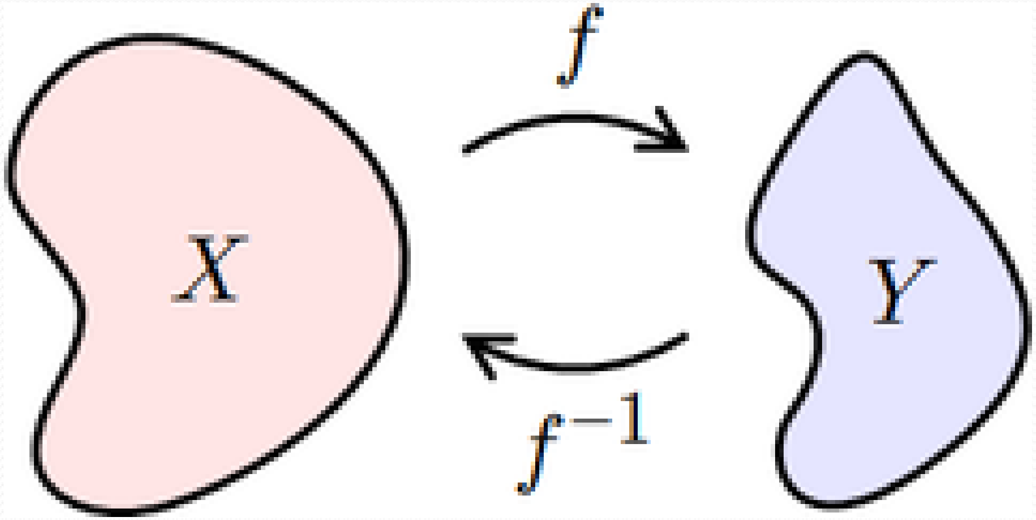
(a) Find $f \circ g$ and determine its domain and range

(b) Find $g \circ f$ and determine its domain and range

Properties for Graph of Functions

- All forms of relations can be represented on coordinates
- To test if a graph displayed is a function, vertical lines are drawn parallel to the y – axis.
- The graph is a function if each vertical line drawn through the domain cuts the graph at only one point.

4. Inverse Function



The Inverse of Functions

- If f is a function, the inverse is denoted by f^{-1}
- Suppose $y=f(x)$ then $x = f^{-1}(y)$

$$y = f(x)$$

$$y = \frac{9}{5}x + 32$$

$$y - 32 = \frac{9}{5}x$$

$$x = \frac{5}{9}(y - 32)$$

$$f^{-1}(y) = \frac{5}{9}(y - 32)$$

Since y could be any variable, we can rewrite f^{-1} as a function of x as

$$f^{-1}(x) = \frac{5}{9}(x - 32)$$

Example 8:

- Find the inverse of :

$$f(x) = \frac{x-3}{2}$$

Graphical Illustration of an Inverse Function

Verify that the inverse of $f(x)$=2x-3 is $f^{-1}(x) = \dfrac{x+3}{2}$

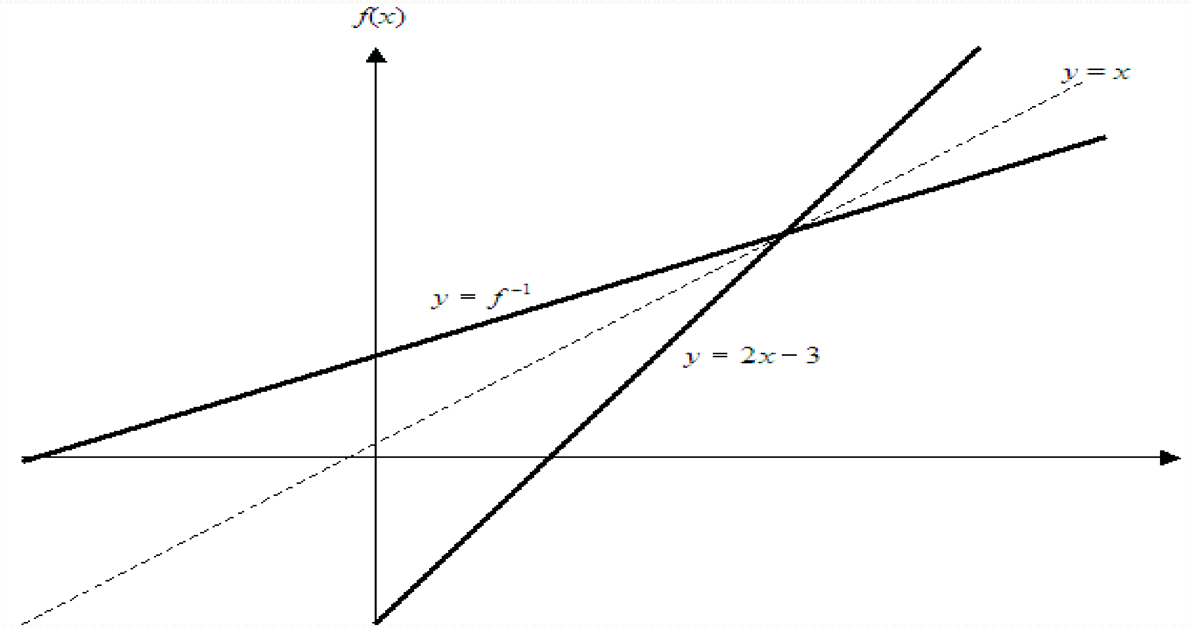
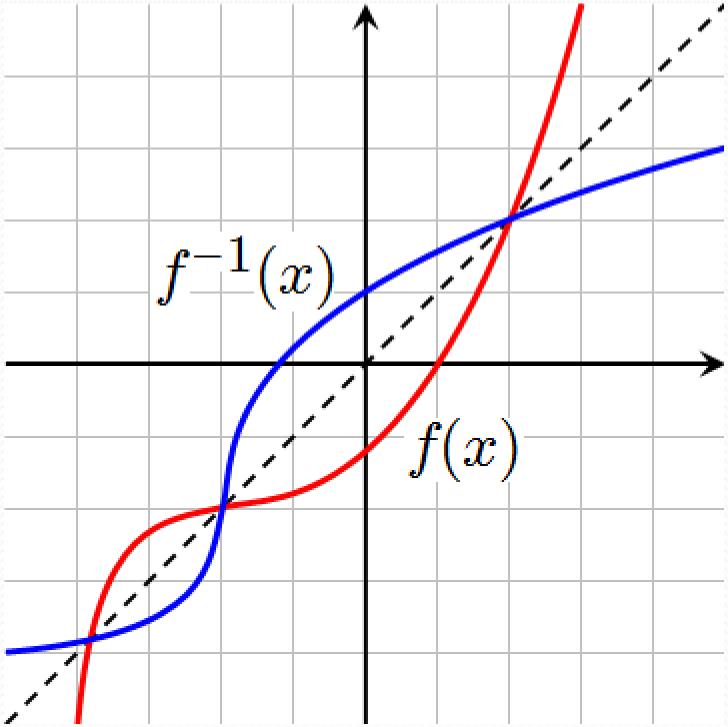
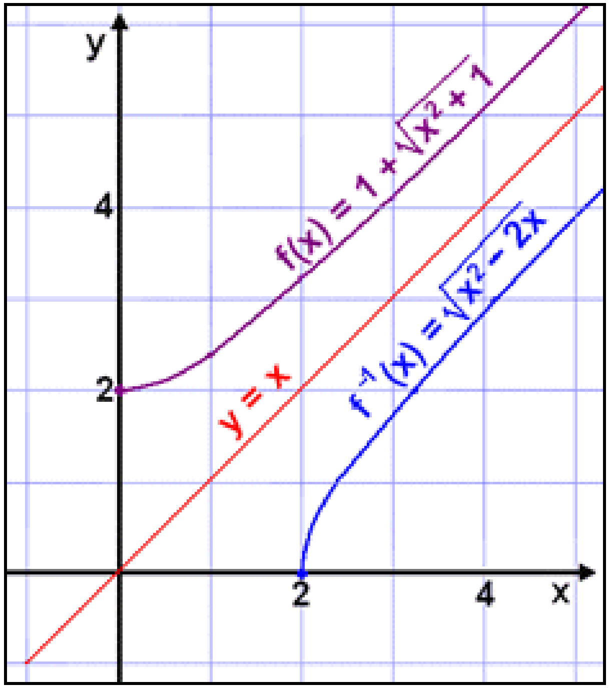


Figure above shows the graph of these two functions on the same pair axes. The dotted line is the graph y=x. These graphs illustrate a general relationship between the graph of a function and that of its inverse, namely that one graph is the reflection of the other in the line $y = x$.



$f^{-1}(x)$
$f(x)$



y
4
2
2
4
x
$f(x) = 1 + \sqrt{x^2 + 1}$
$y = x$
$f^{-1}(x) = \sqrt{x^2 - 2x}$

Example 9:

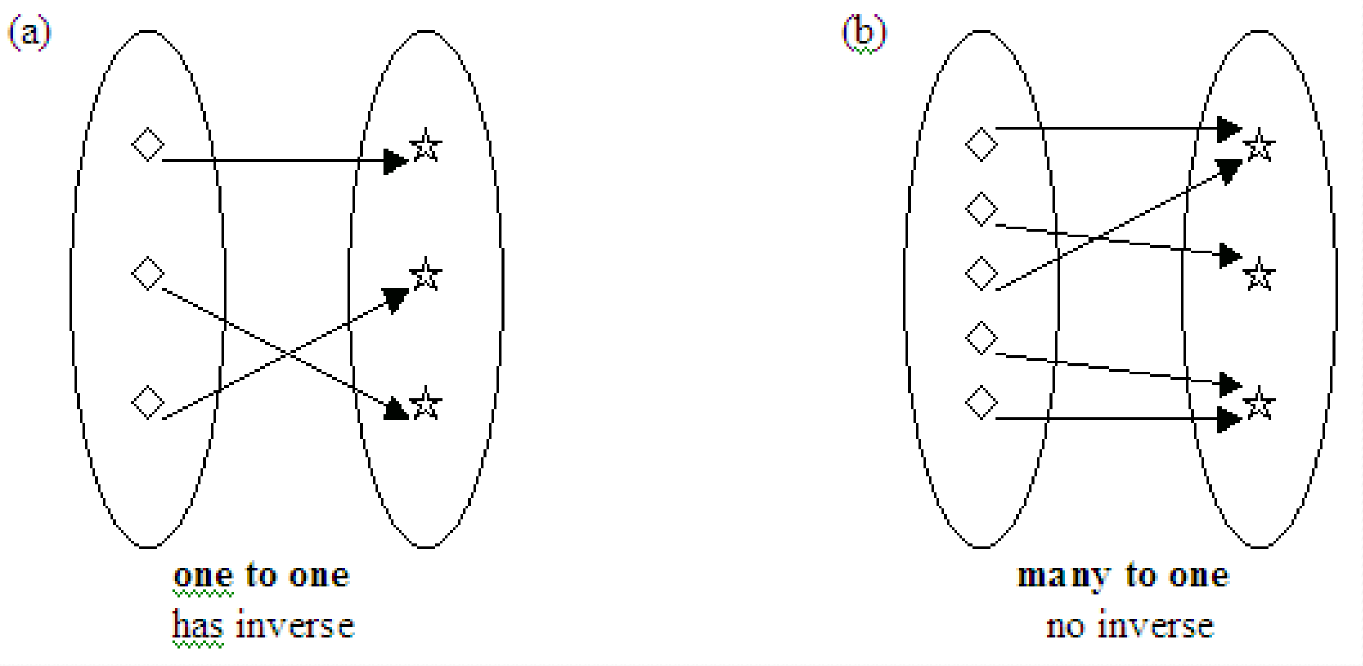
- Find the inverse of :

$$f(x) = \frac{1}{1-x} + 2, x \neq 1.$$

- State the domain of the inverse

FUNCTION WITH NO INVERSE

- An inverse function can only exist if the function is a one-to-one function.

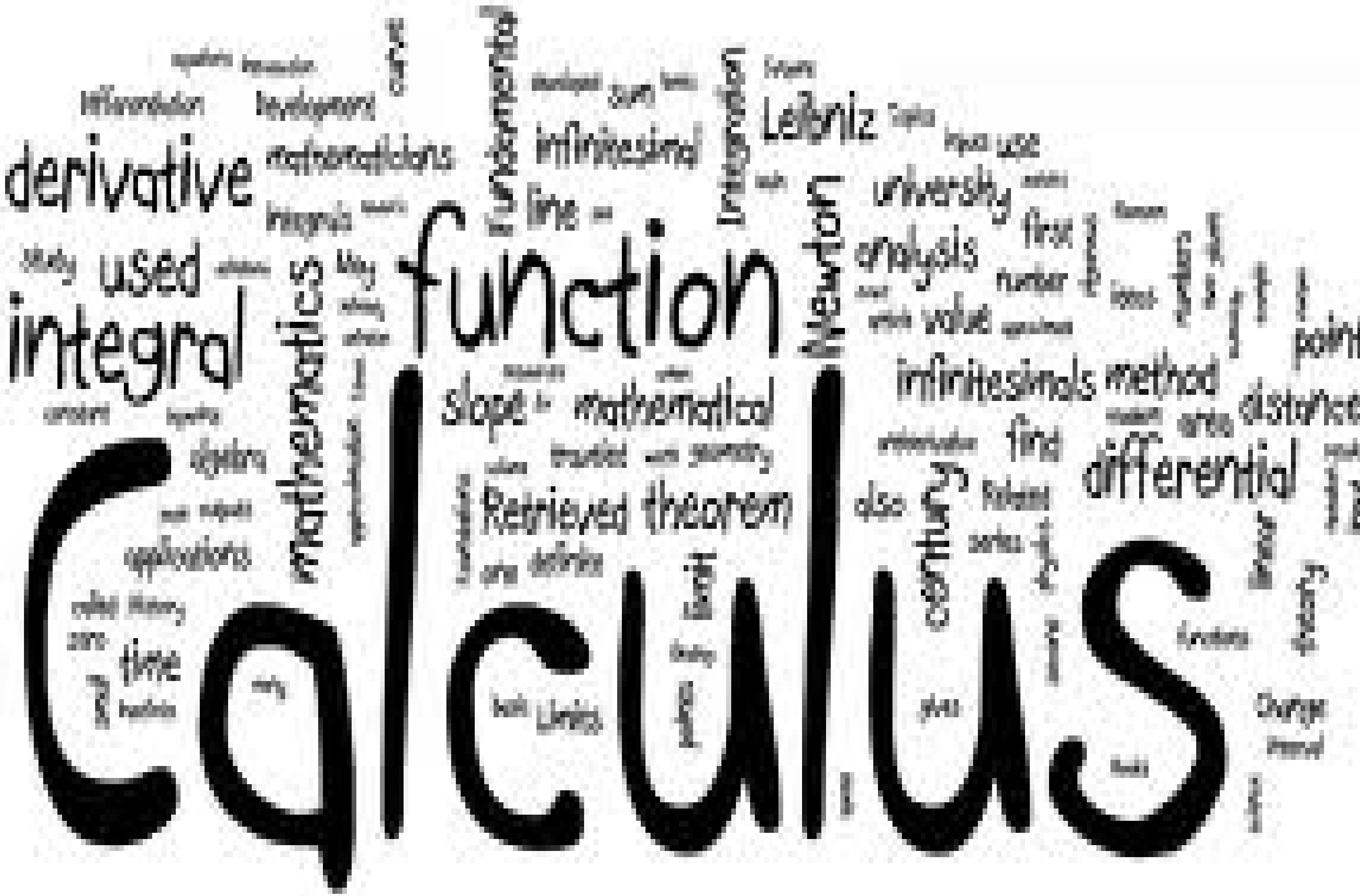


Subtopics

1. **Relations and Functions**
2. **Representation of Functions**
3. **New Function form Old Function**
4. **Inverse of Functions**

Next week lecture:

1. **Exponential Functions** e^x
2. **Logarithm Functions, log** x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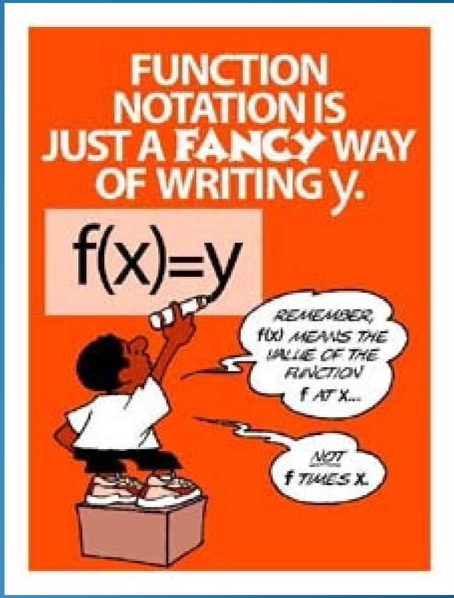
Chapter 1

FUNCTIONS AND GRAPHS



Subtopics

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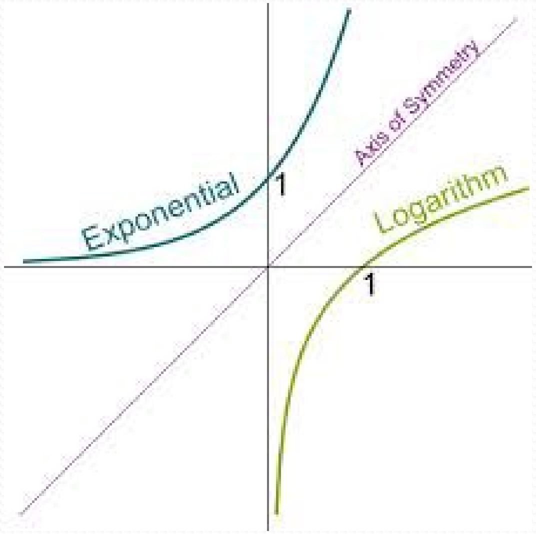
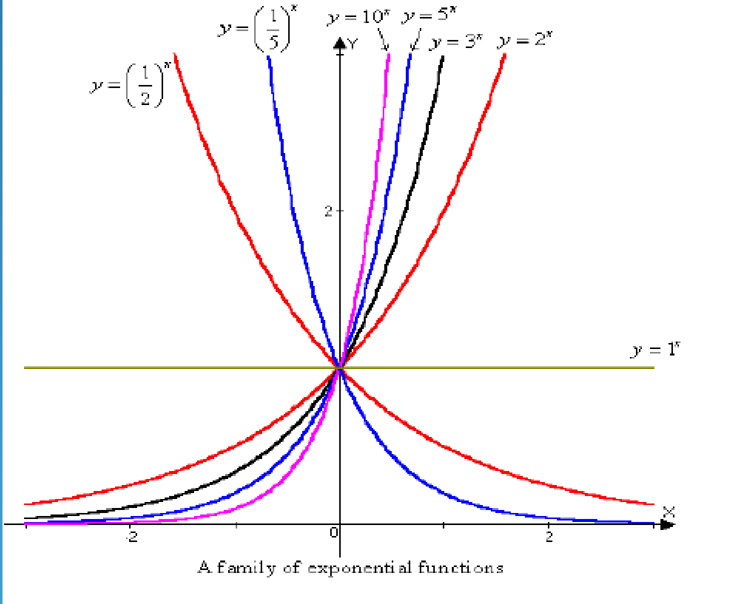
6. **Logarithm Functions, log x**



5. Exponential Functions



Definition: Exponential Functions

- The **exponential function** f with base a, is denoted by

$$f(x) = a^x$$

- where $a > 0, a \neq 1$, and x is any real number
- $f(x) = e^x$ (Natural Exponential Function)

  e = 2.71828...

Evaluation: Exponential Functions

The value of $f(x) = 3^x$ when $x = 2$ is

$$f(2) = 3^2 = \quad 9$$

The value of $f(x) = 3^x$ when $x = -2$ is

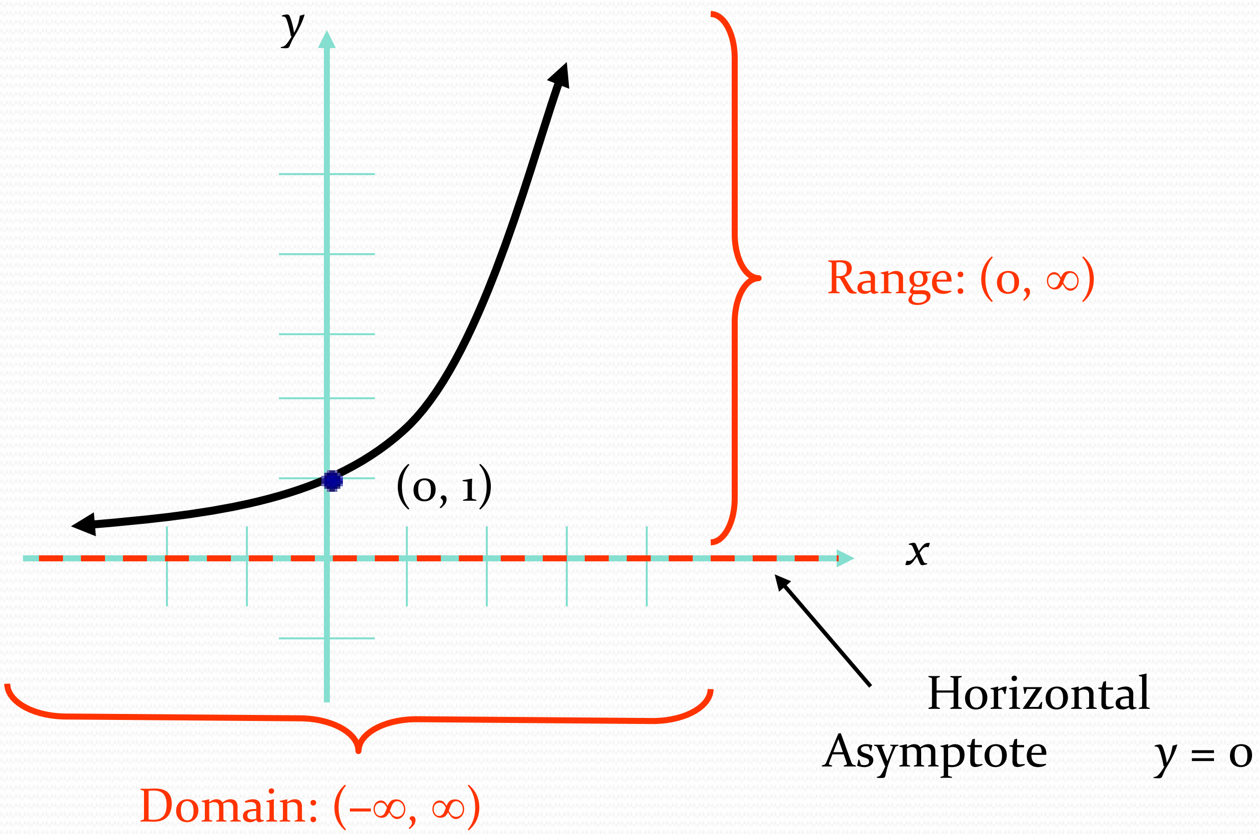
$$f(-2) = 3^{-2} = \frac{1}{9}$$

The value of $g(x) = 0.5^x$ when $x = 4$ is

$$g(4) = 0.5^4 = \quad 0.0625$$

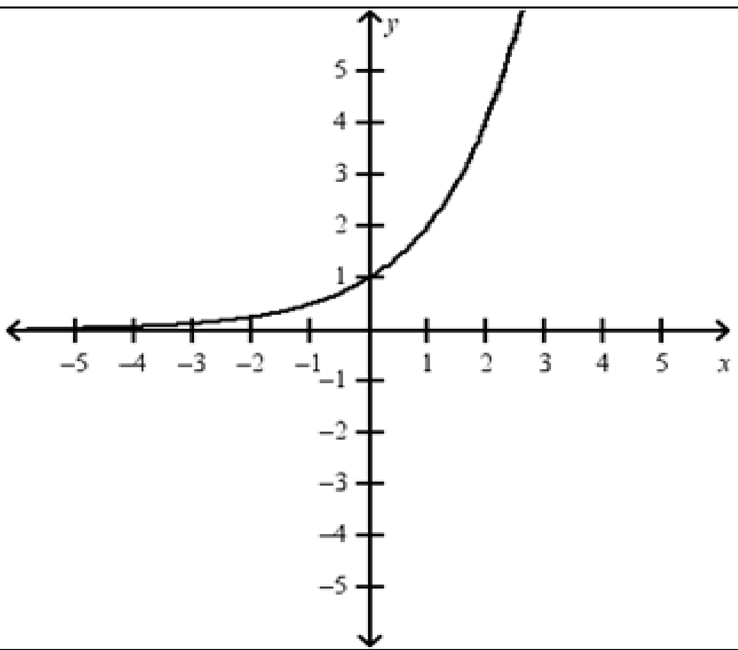
Graph: Exponential Function

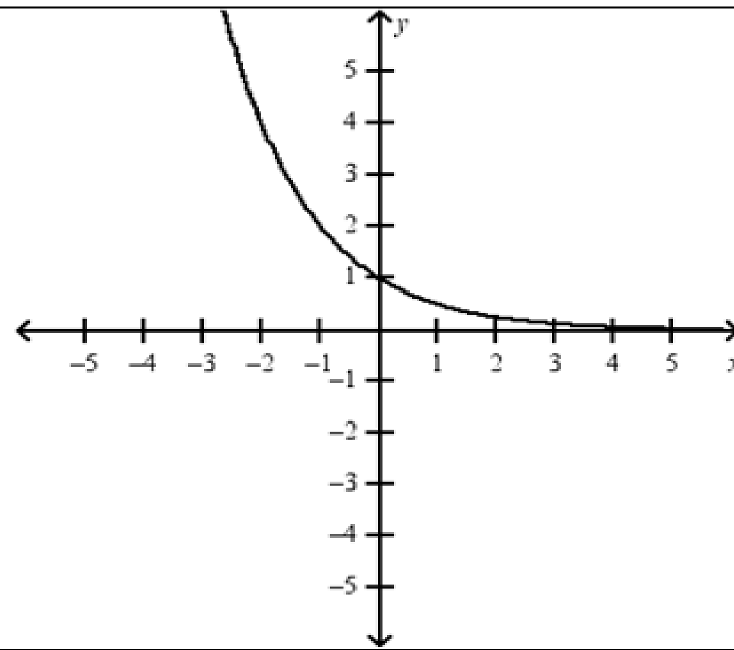
The Graph of $f(x) = a^x$, $a > 1$



Properties: Exponential Function

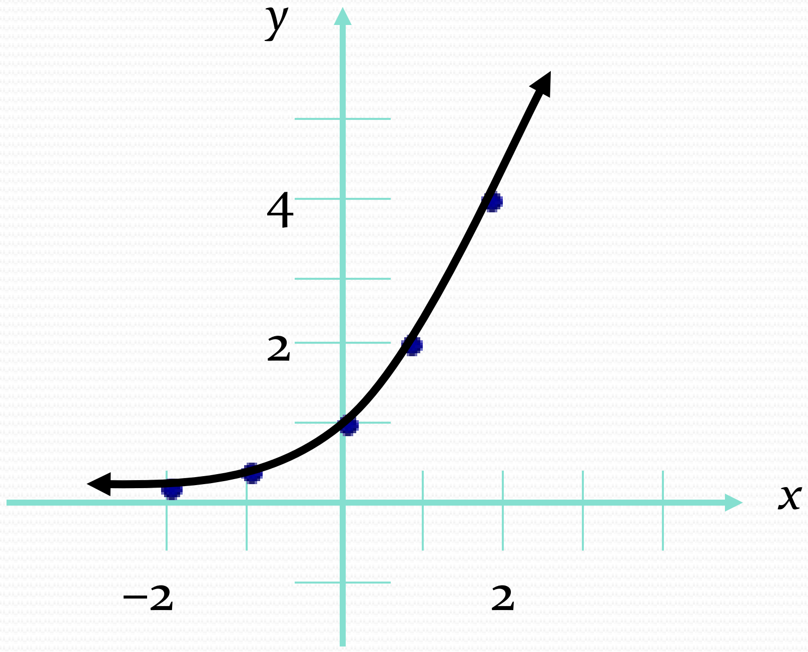
$f(x) = a^x, a > 1$	$f(x) = a^{-x}, \quad a > 1$
Domain: $(-\infty, \ \infty)$	Domain: $(-\infty, \ \infty)$
Range: $(0, \ \infty)$	Range: $(0, \ \infty)$
Intercept: $(0, 1)$	Intercept: $(0, 1)$
Increasing on: $(-\infty, \ \infty)$	Decreasing on : $(-\infty, \ \infty)$
x-axis is a horizontal asymptote	x-axis is a horizontal asymptote
continuous	continuous





Example 1: Sketch the graph of $f(x) = 2^x$.

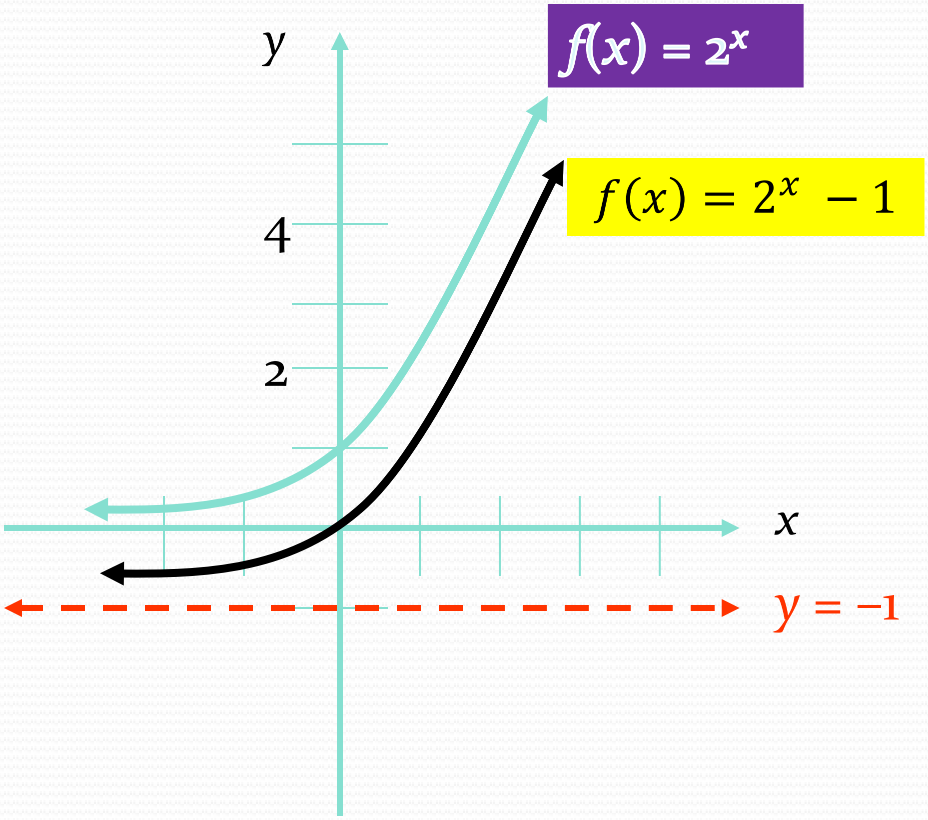
x	$f(x)$	$(x, f(x))$
-2	¼	(-2, ¼)
-1	½	(-1, ½)
0	1	(0, 1)
1	2	(1, 2)
2	4	(2, 4)



Example 2:

Sketch the graph of $g(x) = 2^x - 1$. State the domain and range.

The graph of this function is a vertical translation of the graph of $f(x) = 2^x$ down one unit



Domain: $(-\infty, \infty)$

Range: $(-1, \infty)$

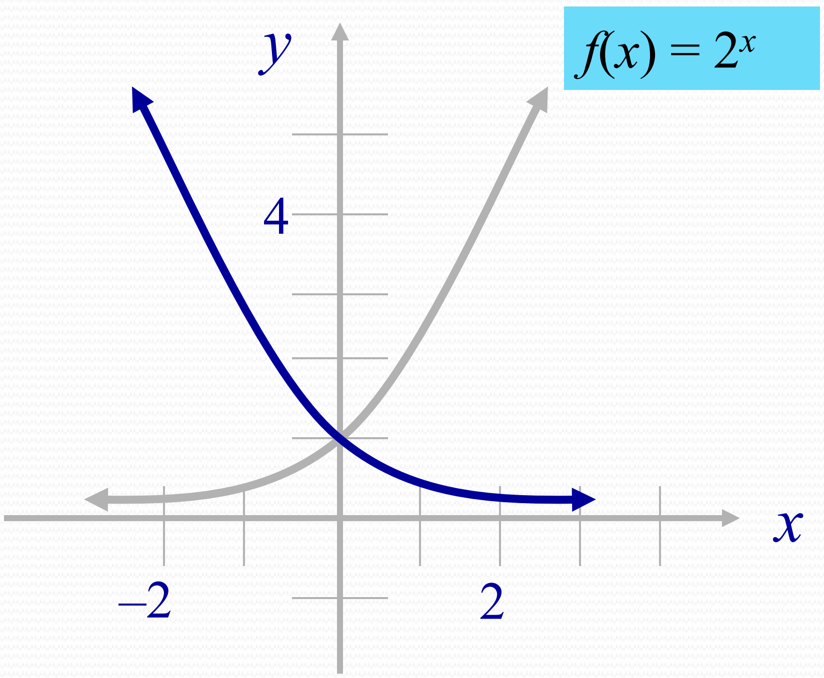
Example 3:

The graph of this function $f(x) = 2^{-x}$

is a reflection the graph of $f(x) = 2^x$ in the y-axis.

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$



*Rules: Exponential Functions

(All laws apply for any positive a, b, x, and y.)	
$x = b^y$ is the same as $y = \log_b x$	
$b^0 = 1$	$b^1 = b$
$b^{\log_b x} = x$	
$(b^x)^y = b^{xy}$	$b^x\, b^y = b^{x+y}$
$\frac{b^x}{b^y} = b^{x-y}$	$b^{-x} = \frac{1}{b^x}$
$b^{\frac{x}{y}} = \sqrt[y]{b^x} = (\sqrt[y]{b})^x$	$\sqrt[x]{a}\sqrt[x]{b} = \sqrt[x]{ab}$

*Example 4:

Simplify and evaluate if possible

$$x^3 x^5$$

$$(g^3)^2$$

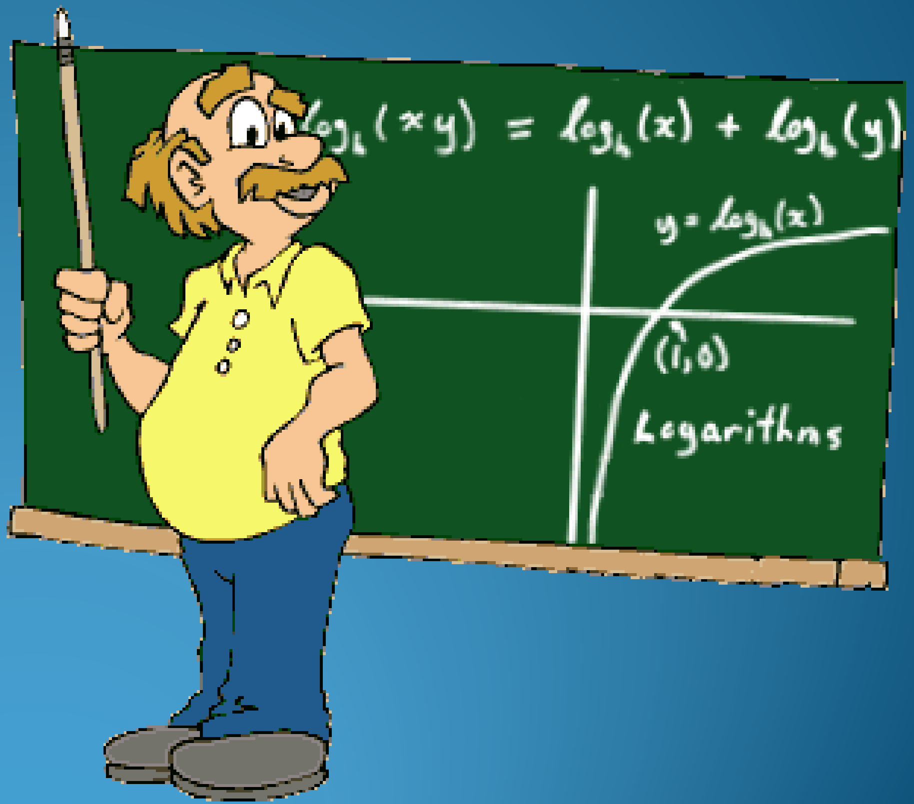
$$e^0$$

$$8^{\frac{2}{3}}$$

$$\left(\frac{16}{81}\right)^{-\frac{3}{4}}$$

$$\frac{p^2}{p^6}$$

6. Logarithmic Functions



*Definition: Logarithmic Functions

- For $x > 0$ and $0 < a \neq 1$,

$$y = \log_a x \text{ if and only if } x = a^y.$$

- The function given by $f(x) = \log_a x$ is called the **logarithmic function with base *a***.

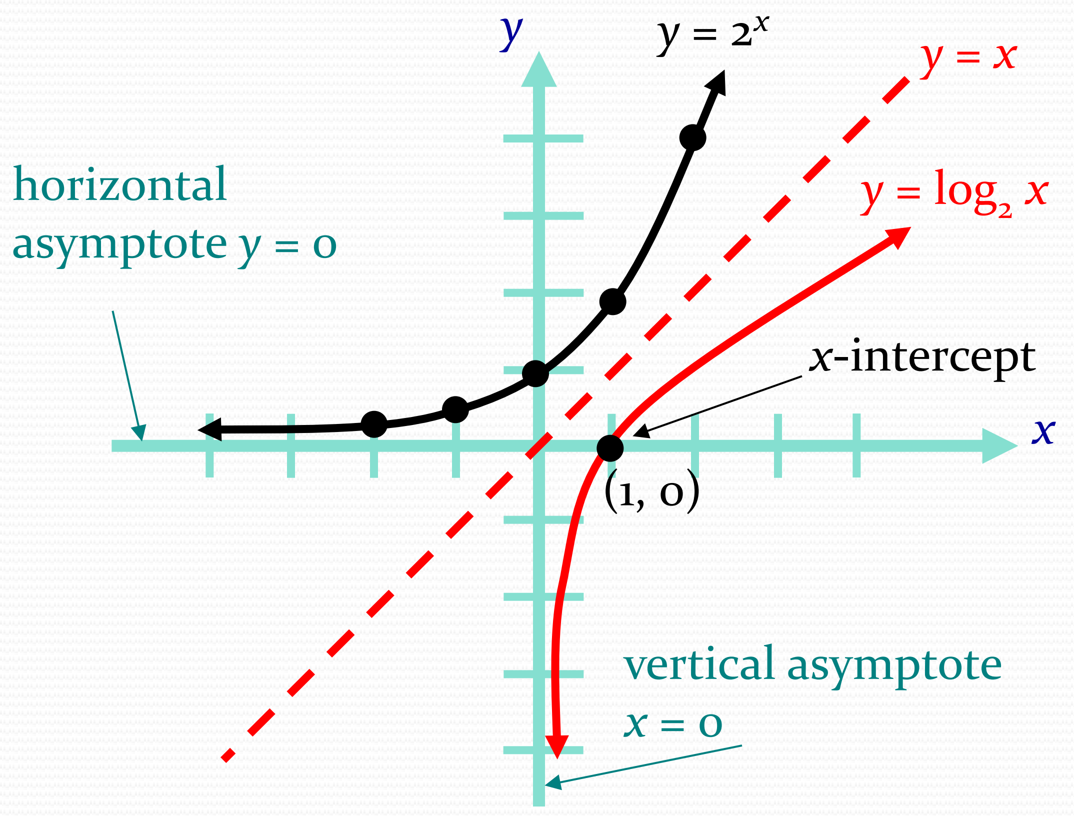
- $\log_{10}$ x = log x (Common log)
- $\log_e$ x = ln x (Natural log)

  e = 2.71828...

Relationship: $(y = b^x)$ Vs $(y = \log_b x)$

Graph $f(x) = \log_2 x$ and $f(x) = 2^x$

The logarithm function is the *inverse* of the exponential function of the same base, ∴ its graph is the reflection of the exponential function in the line $y = x$.



Relationship: $(y = b^x)$ Vs $(y = \log_b x)$

Every logarithmic equation has an equivalent exponential form:

$y = \log_a x$ is equivalent to $x = a^y$

A logarithm is an exponent!

A logarithmic function is the inverse function of an exponential function.

Exponential function: $y = a^x$

Logarithmic function: $y = \log_a x$ is equivalent to $x = a^y$

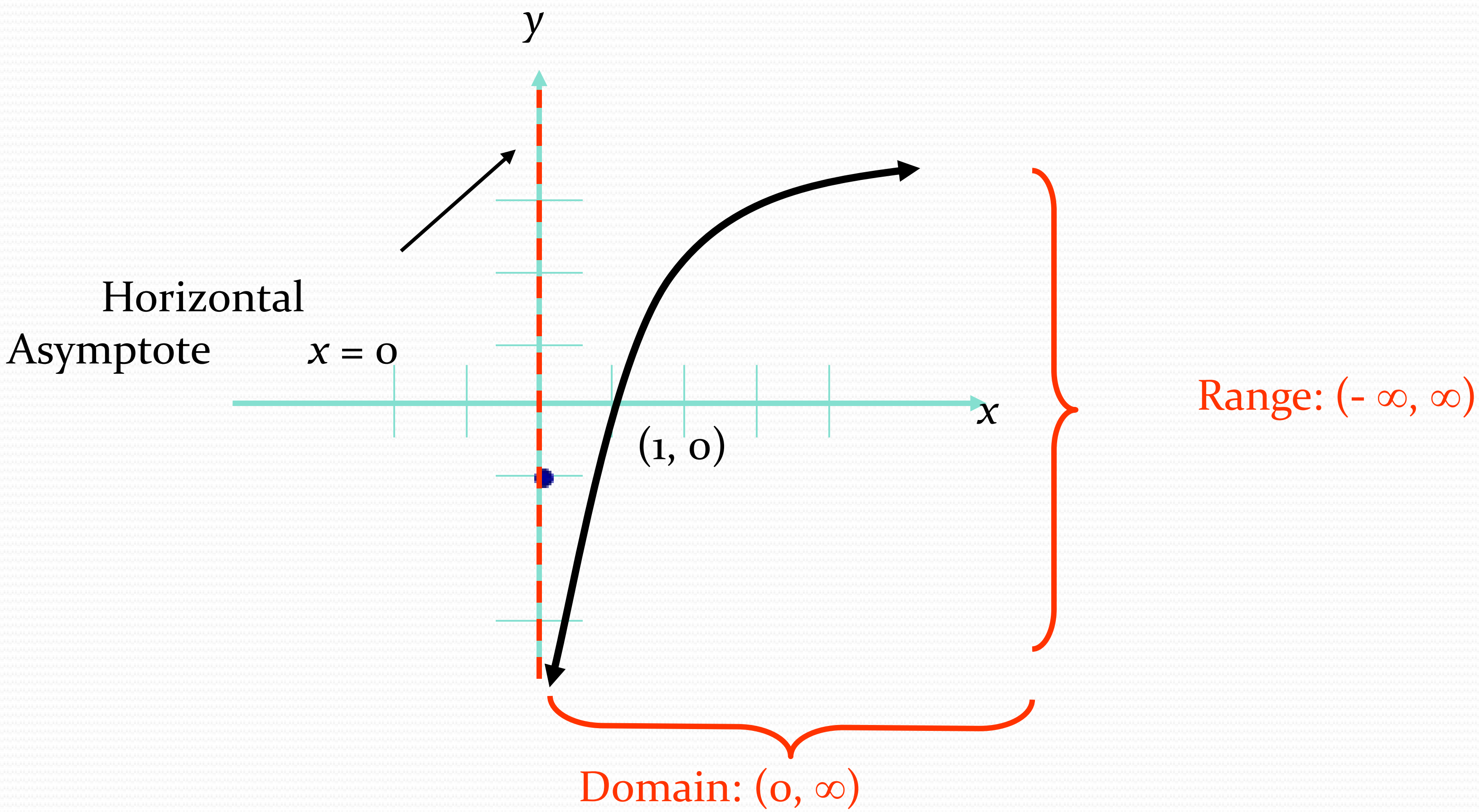
*Example 5:

Exponential Form	Logarithmic Form
$2^3 = 8$	$\log_2 8 = 3$
$\left(\frac{1}{2}\right)^{-4} = 16$	$\log_{1/2} 16 = -4$
$5^1 = 5$	$\log_5 5 = 1$
$\left(\frac{3}{4}\right)^0 = 1$	$\log_{3/4} 1 = 0$

Logarithmic function: $y = \log_a x$ is equivalent to $x = a^y$

Graph: Logarithmic Function

The Graph of $f(x) = a^x$, $a > 1$



Properties: Logarithmic Function

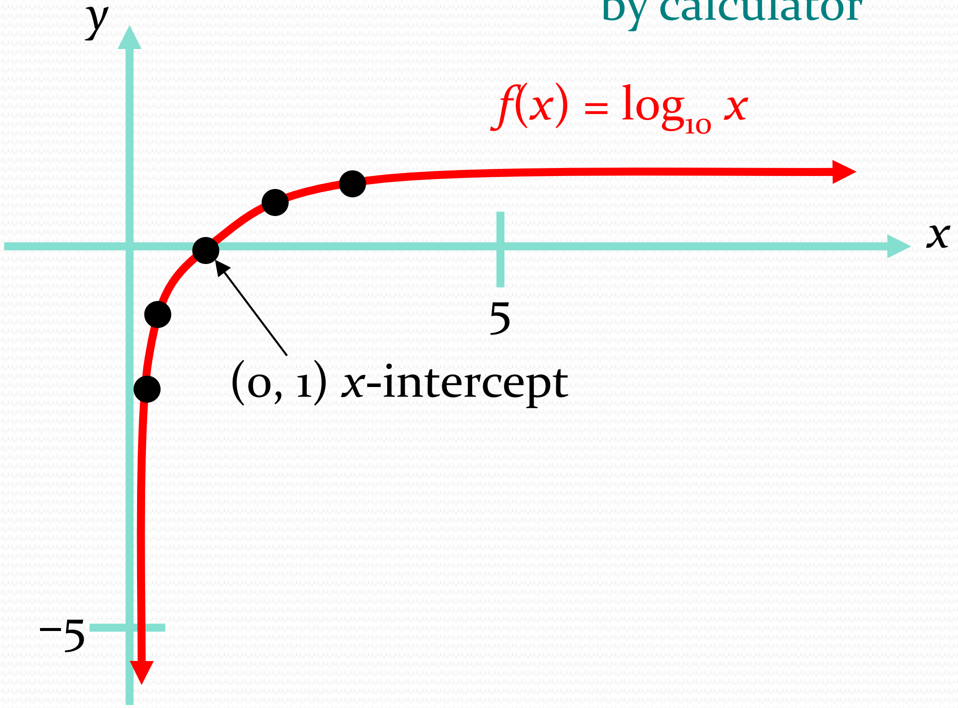
1. Domain: $(0, \infty)$
2. Range: $(-\infty, \infty)$
3. x-intercept: $(1, 0)$
4. Continuous on $(0, \infty)$
5. Increasing on $(0, \infty)$ if $a > 1$
6. Decreasing on $(0, \infty)$ if $a < 1$

Example 6:

Graph the common logarithm function $f(x) = \log_{10} x$.

x	$\frac{1}{100}$	$\frac{1}{10}$	1	2	4	10
$f(x) = \log_{10} x$	–2	–1	0	0.301	0.602	1

by calculator



*Rules: Logarithmic Functions

1. $\log_a 1 = 0$ since $a^0 = 1$.
2. $\log_a a = 1$ since $a^1 = a$.
3. $\log_a a^x = x$ and $a^{\log_a x} = x$ — inverse property
4. If $\log_a x = \log_a y$, then $x = y$. — one-to-one property
5. $\log_a xy = \log_a x + \log_a y$ — Product rule
6. $\log_a x/y = \log_a x - \log_a y$ — Quotient rule
7. $\log_a x^y = y\log_a x$ — Power Rule

Change-of-Base Rule

For any positive real numbers x, a, and b, where $a \neq 1$ and $b \neq 1$,

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

*Example 7:

Solve for x: $\log_6 6 = x$

$$\log_6 6 = 1$$

Simplify: $\log_3 3^5$

$$\log_3 3^5 = 5$$

Simplify: $7^{\log_7 9}$

$$7^{\log_7 9} = 9$$

Simplify: $\log_5 \sqrt{8}$

$$= \log_5 8^{½} = \frac{1}{2} \log_5 8$$

*Example 8:

1. Simplify, $\log 7 + \log 4 - \log 2$

$\log \frac{7*4}{2} = \log 14$

2. Simplify, $\ln e^2$

$= 2 \ln e$

$= 2 \log_e e = 2 * 1 = 2$

3. Simplify, $e^{4 \ln 3 - 3 \ln 4}$

$= e^{\ln 3^4 - \ln 4^3}$

$= e^{\ln 81/64}$

$= e^{\log_e 81/64} = 81/64$

*Rules of Natural Logarithms

1. $\ln 1 = 0$ since $e^0 = 1$.
2. $\ln e = 1$ since $e^1 = e$.
3. $\ln e^x = x$ and $e^{\ln x} = x$ inverse property
4. If $\ln x = \ln y$, then $x = y$. one-to-one property

Simplify each expression.

$$\ln\left(\frac{1}{e^2}\right) = \ln\left(e^{-2}\right) = -2$$ inverse property

$$e^{\ln 20} = 20$$ inverse property

$$3\ln e = 3(1) = 3$$ property 2

$$\sqrt{\ln 1} = \sqrt{0} = 0$$ property 1

Example 9 :

Write the equivalent exponential equation and solve for *y*.

Logarithmic Equation	Equivalent Exponential Equation	Solution
$y = \log_2 16$	$16 = 2^y$	$16 = 2^4 \rightarrow y = 4$
$y = \log_2(\frac{1}{2})$	$\frac{1}{2} = 2^y$	$\frac{1}{2} = 2^{-1} \rightarrow y = -1$
$y = \log_4 16$	$16 = 4^y$	$16 = 4^2 \rightarrow y = 2$
$y = \log_5 1$	$1 = 5^y$	$1 = 5^0 \rightarrow y = 0$

The Equality Property = Exponential & Logarithmic Equations

1. If $\log_b m = \log_b n$, then $m = n$.

2. If $b^m = b^n$, then $m = n$.

Example 10:

1. $\log_6 2x = \log_6(x + 3)$,

$2x = x + 3$

$x = 3.$

(Since the bases are the same we
simply set the exponents equal.)

2. $5^{1-x} = 5^{-2x}$,

$1 - x = -2x$

$x = -1.$

Example 11:

$$3^{2x-5} = 3^{x+3}$$

$$2x - 5 = x + 3$$

$$x - 5 = 3$$

$$x = 8$$

(Since the bases are the same we
simply set the
exponents equal.)

*Solve: Exponential Equation

1. Isolate the base-exponent term.
2. Write as a log. Solve for the variable.

Example 12:

$$4^{x+3} = 7$$

$$\log_4 7 = x + 3$$

$$-3 + \log_4 7 = x$$

OR with change of bases:

$$x = -3 + \frac{\log 7}{\log 4}$$

Another method is to take the LOG of both sides.

Solve: Logarithmic Equation

- Isolate to a single log term.
- Convert to an exponent.
- Solve equation.

Example 13:

$\log x + \log (x - 15) = 2$

$\log x(x - 15) = 2$

so $10^2 = x\,(x - 15)$

$100 = x^2 - 15x$

$0 = x^2 - 15x - 100$

$0 = (x - 20)\,(x + 5)$

$\therefore x = 20 \text{ or } -5$

Applications of logarithmic and exponential functions

Example 14:

A normal child's systolic blood pressure may be approximated by the function $p(x) = m(\ln x) + b$ where $p(x)$ is measured in millimeters of mercury, x is measured in pounds, and m and b are constants. Given that m = 19.4 and b = 18, determine the systolic blood pressure of a child who weighs 92 lb.

Since $m = 19.4$, $x = 92$, and $b = 18$

we have $p(92) = 19.4(\ln 92) + 18$

$$= 105.72$$

Example 15:

The formula $R = \frac{1}{10^{12}} e^{\frac{-t}{8223}}$ (t in years) is used to estimate the age of organic material. The ratio of carbon 14 to carbon 12 in a piece of charcoal found at an archaeological dig is $R = \frac{1}{10^{15}}$. How old is it?

$$\frac{1}{10^{12}} e^{\frac{-t}{8223}} = \frac{1}{10^{15}}$$ original equation

$$e^{\frac{-t}{8223}} = \frac{1}{1000}$$ multiply both sides by 10^{12}

$$\ln e^{\frac{-t}{8223}} = \ln \frac{1}{1000}$$ take the natural log of both sides

$$\frac{-t}{8223} = \ln \frac{1}{1000}$$ inverse property

$$t = -8223\left(\ln \frac{1}{1000}\right) \approx -8223(-6.907) = 56796$$

To the nearest thousand years the charcoal is 57,000 years old.

Subtopics

1. **Relations and Functions**
2. **Representation of Functions**
3. **New Function form Old Function**
4. **Inverse of Functions**
5. **Exponential Functions** e^x
6. **Logarithm Functions, log** x

Chapter 2

Limits and Continuity

Outlines:

- **Limit**
  - **Definition Of Limit**
  - **Properties Of Limits**
  - **Limit Of Infinite Function**
  - **Define Limits At Infinity**
- **Continuity**
  - **The Continuity Test**

Limits

The word "limit" is used in everyday conversation to describe the ultimate behavior of something, as in the "limit of one's endurance" or the "limit of one's patience."

In mathematics, the word "limit" has a similar but more precise meaning.

Limits

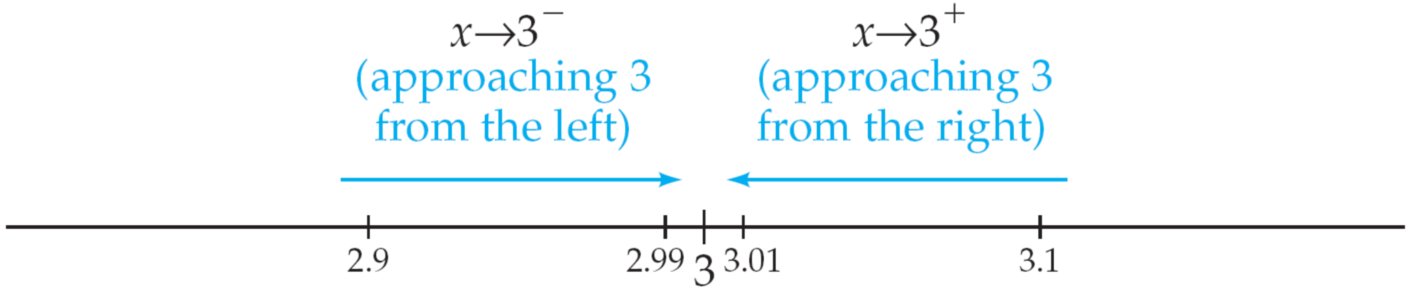
Given a function *f*(*x*), if *x* approaching 3 causes the function to take values approaching (or equaling) some particular number, such as 10, then we will call 10 *the limit of the function* and write

$$\lim_{x \to 3} f(x) = 10$$ Limit of $f(x)$ as x approaches 3 is 10

In practice, the two simplest ways we can approach 3 are *from the left* or *from the right.*

Limits

For example, the numbers 2.9, 2.99, 2.999, ... approach 3 *from the left*, which we denote by $x \to 3^-$, and the numbers 3.1, 3.01, 3.001, ... approach 3 *from the right*, denoted by $x \to 3^+$. Such limits are called *one-sided limits.*



Example 1 – Finding A Limit By Tables

Use tables to find

$$\lim_{x \to 3} (2x + 4).$$ Limit of $2x + 4$ as x approaches 3

Solution :

We make two tables, as shown below, one with x approaching 3 *from the left*, and the other with x approaching 3 *from the right.*

Approaching 3 *from the left* ↓

x	$2x + 4$
2.9	9.8
2.99	9.98
2.999	9.998

Limit is 10 ↓

This table shows $\lim_{x \to 3^-} (2x + 4) = 10$

Approaching 3 *from the right* ↓

x	$2x + 4$
3.1	10.2
3.01	10.02
3.001	10.002

Limit is 10 ↓

This table shows $\lim_{x \to 3^+} (2x + 4) = 10$

Limits- Definition

- A function ***f*(*x*)** is said to approach a constant ***L*** as a limit when ***x*** approaches ***a*** as below,

$$\lim_{x \to a} f(x) = L$$

- Where *f*(*x*) is the function which assumes a corresponding set of values and ***x*** is independent variables.

Limit- Exist

Thus we have a left-sided limit: $\lim_{x \to c^-} f(x) = K$

And a right-sided limit: $\lim_{x \to c^+} f(x) = L$

And in order for a limit to **EXIST**, the limit from the left and the limit from the right must exist and be equal.

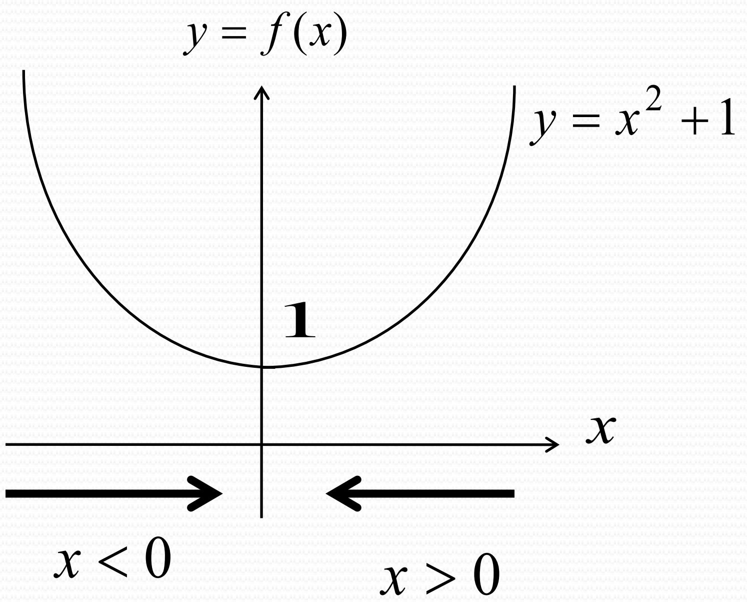
Example 2: Find $\lim_{x \to 0}(x^2 + 1)$

x approaching 0 from the left → ← x approaching 0 from the right

x	-1	-0.5	-0.25	-0.001	0	0.001	0.25	0.5	1
f(x)	2	1.25	1.0625	1.00001	?	1.00001	1.0625	1.25	2

The graph of

$$y = f(x) = x^2 + 1$$



Example 2: Find $\lim_{x \to 0}(x^2 + 1)$

The closer ***x*** gets to the values 0, the closer the value of ***f*(*x*)** comes to 1.
This limit could have been determined by simple substituting $x=0$ into ***f*(*x*)**.

$$\lim_{x \to 0^-}(x^2 + 1) = \lim_{x \to 0^+}(x^2 + 1) = 1$$

Example 3 –
Finding Limits By Direct Substitution

As you have just seen the good news is that many limits can be evaluated by *direct substitution*.

1. $\lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$ Substitute 4 for x.

2. $\lim_{x \to 6} \dfrac{x^2}{x + 3} = \dfrac{6^2}{6 + 3} = \dfrac{36}{9} = 4$ Substitute 6 for x.

PROPERTIES OF LIMITS

Let a be a real number, and suppose that $\lim_{x \to a} f(x) = \mathrm{h}$ and $\lim_{x \to a} g(x) = \mathrm{k}$

then

$$\lim_{x \to a} \left[f(x) \pm g(x) \right] = \mathrm{h} \pm \mathrm{k}$$

$h > 0)$

$$\lim_{x \to a} \left[f(x) g(x) \right] = \mathrm{hk}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\mathrm{h}}{\mathrm{k}} \quad (if \qquad k \neq 0)$$

$$\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n = a^n \qquad (\text{if n} > 0)$$

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{h} \qquad (if \ n \, even \quad h > 0)$$

Examples 4: Using Limit properties

1. $$\lim_{x\to 3}\left(x^2+1\right)=\lim_{x\to 3}x^2+\lim_{x\to 3}1$$
$$=\left(\lim_{x\to 3}x\right)^2+\lim_{x\to 3}1$$
$$=3^2+1=10$$

2. $$\lim_{x\to 1}\frac{\sqrt{2x-1}}{3x+5}=\frac{\sqrt{\lim_{x\to 1}(2x-1)}}{\lim_{x\to 1}(3x+5)}=\frac{\sqrt{2\lim_{x\to 1}x-\lim_{x\to 1}1}}{3\lim_{x\to 1}x+\lim_{x\to 1}5}$$
$$=\frac{\sqrt{2-1}}{3+5}=\frac{1}{8}$$

Limit for case : $\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{0}{0}$

1. **<u>Factorization method</u>**
   When this case happens we can factorize the numerator and denominator and simplify the fraction.

2. **<u>Multiplication of Conjugates method</u>**
   When this case happens we can find the conjugate of the function.

Example 5 : **Factorization method**

$$\lim_{x\to3}\frac{x^2-9}{x-3}=\frac{\lim\limits_{x\to3}x^2-9}{\lim\limits_{x\to3}x-3}=\frac{0}{0}$$

$$\lim_{x\to3}\frac{x^2-9}{x-3}=\lim_{x\to3}\frac{(x-3)(x+3)}{x-3}$$

$$\lim_{x\to3}(x+3)=6$$

Example 6 : **Multiplication of Conjugates method**

$$\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}=\frac{\lim\limits_{x\to 0}\sqrt{x^2+9}-3}{\lim\limits_{x\to 0}x^2}=\frac{0}{0}$$

$$\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}=\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}\cdot\frac{\sqrt{x^2+9}-3}{\sqrt{x^2+9}-3}$$

$$=\lim_{x\to 0}\frac{(x^2+9)-9}{x^2(\sqrt{x^2+9}+3)}=\lim_{x\to 0}\frac{x^2}{x^2(\sqrt{x^2+9}+3)}$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{x^2+9}+3}=\frac{1}{3+3}=\frac{1}{6}$$

LIMIT OF INFINITE FUNCTION

$$\lim_{x \to a} f(x) = +\infty$$

$$\lim_{x \to a} f(x) = -\infty$$

$$\lim_{x \to 0^+} \left(\frac{1}{x^r} \right) = +\infty$$

$$\lim_{x \to 0^-} \left(\frac{1}{x^r} \right) = \begin{cases} -\infty & r \quad odd\ numbers \\ +\infty & r \quad even\ numbers \end{cases}$$

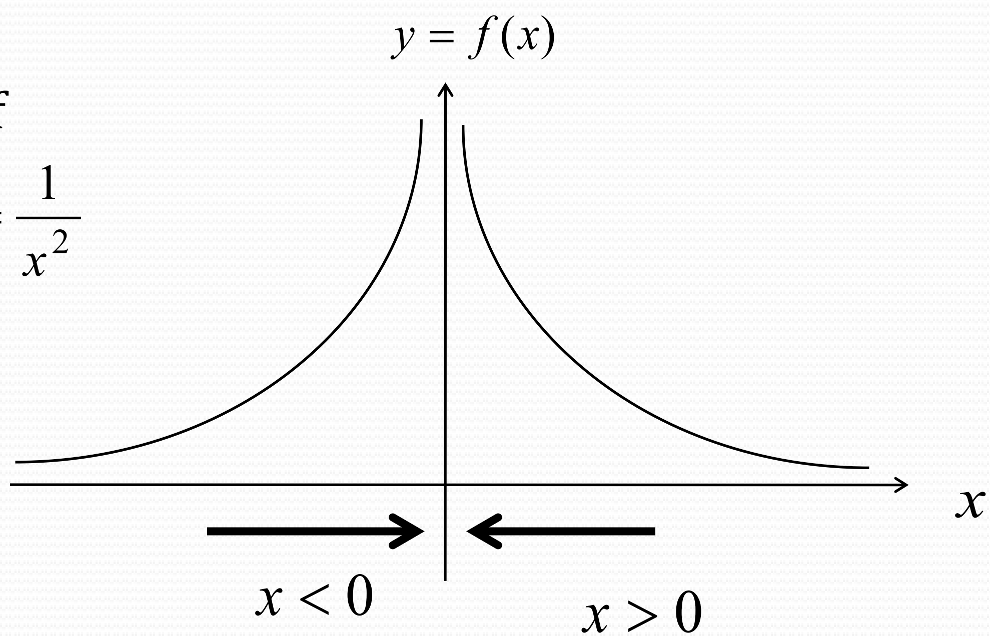
Example 7: Find $\lim_{x\to 0}\left(\frac{1}{x^2}\right)$

x approaching 0 from the left → ← x approaching 0 from the right

x	-1	-0.5	-0.1	-0.001	0	0.001	0.1	0.5	1
f(x)	1	4	100	1000000	?	1000000	100	4	1

The graph of

$$y = f(x) = \frac{1}{x^2}$$



We should conclude that

$$\lim_{x\to 0}\left(\frac{1}{x^2}\right) = +\infty$$

Example 8:

Given $f(x)=\dfrac{1}{x-1}$. Find $\lim\limits_{x\to 1^+} f(x)$ and $\lim\limits_{x\to 1^-} f(x)$

DEFINE LIMITS AT INFINITY

Theorem

(a) If $\lim_{x \to \infty} f(x) = a$ and $\lim_{x \to \infty} f(x) = b$, then $a = b$.

(b) For any positive integer n,

i) $\lim_{x \to \infty} \left[\frac{1}{x^n} \right] = 0$

ii) $\lim_{x \to -\infty} \left[\frac{1}{x^n} \right] = 0$

(c) If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$

Divide each term by x to the highest power of the denominator.

Example 9:

Find the limit of $\lim_{x\to\infty} \frac{2x^3+x^2-3}{x^3+x+2}$

$$\lim_{x\to\infty} \frac{\frac{2x^3}{x^3}+\frac{x^2}{x^3}-\frac{3}{x^3}}{\frac{x^3}{x^3}+\frac{x}{x^3}+\frac{2}{x^3}}$$

$$\lim_{x\to\infty} \frac{2+\frac{1}{x}-\frac{3}{x^3}}{1+\frac{1}{x^2}+\frac{2}{x^3}}$$

$$\lim_{x\to\infty} \frac{2x^3+x^2-3}{x^3+x+2} = \frac{2+0-0}{1+0+0} = 2$$

2. $$\lim_{x\to\infty}\left(\frac{4x^2-5x+21}{7x^3+5x^2-10x+1}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\frac{4x^2}{x^3}-\frac{5x}{x^3}+\frac{21}{x^3}}{\frac{7x^3}{x^3}+\frac{5x^2}{x^3}-\frac{10x}{x^3}+\frac{1}{x^3}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\frac{4}{x}-\frac{5}{x^2}+\frac{21}{x^3}}{7+\frac{5}{x}-\frac{10}{x^2}+\frac{1}{x^3}}\right)$$

$$=\frac{0}{7}$$

$$=0$$

3. $$\lim_{x\to\infty}\left(\frac{x^2+2x-4}{12x+31}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\frac{x^2}{x}+\frac{2x}{x}-\frac{4}{x}}{\frac{12x}{x}+\frac{31}{x}}\right)$$

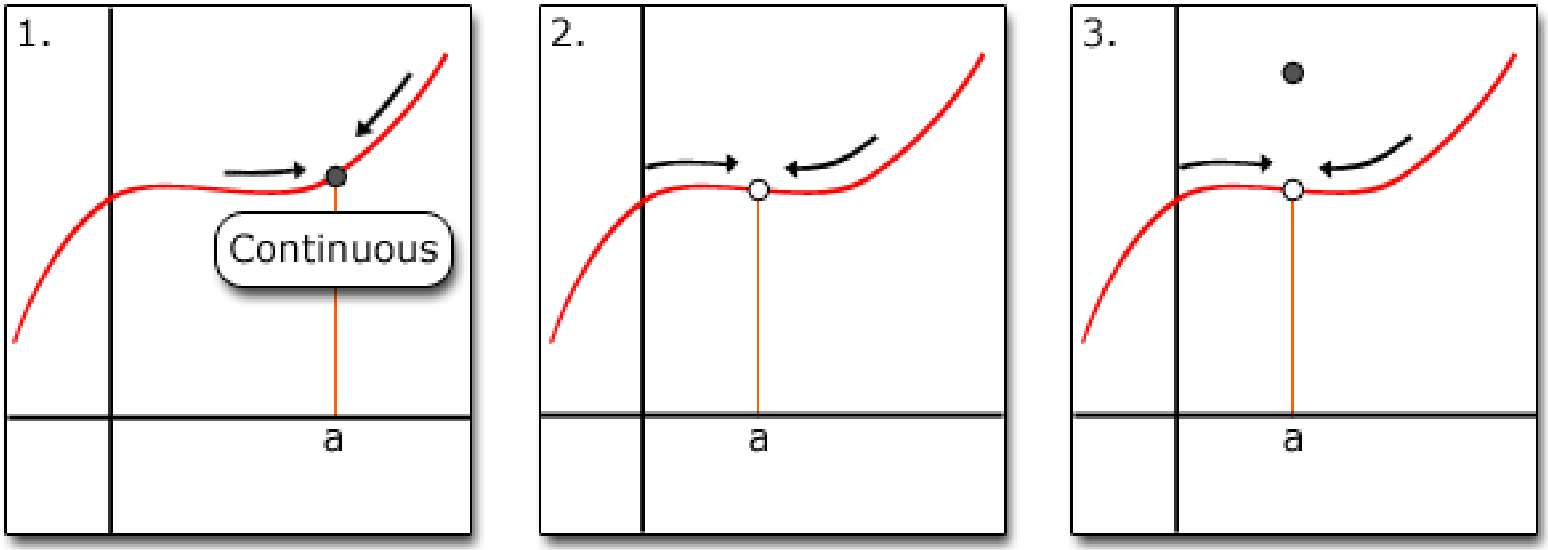
$$=\lim_{x\to\infty}\left(\frac{x+2-\frac{4}{x}}{12+\frac{31}{x}}\right)$$

$$=\frac{\infty+2}{12}$$

$$=\infty$$

CONTINUITY

- A function $y = f(x)$ that can be graphed throughout its domain with one continuous motion of the pen (that is, without lifting the pen) is an example of a **continuous** function.



The Continuity Test:

A function f is <u>continuous at a point</u> x = c if

1. f (c) is defined

2. $\lim_{x \to c} f(x)$ exists

3. $\lim_{x \to c} f(x) = f(c)$

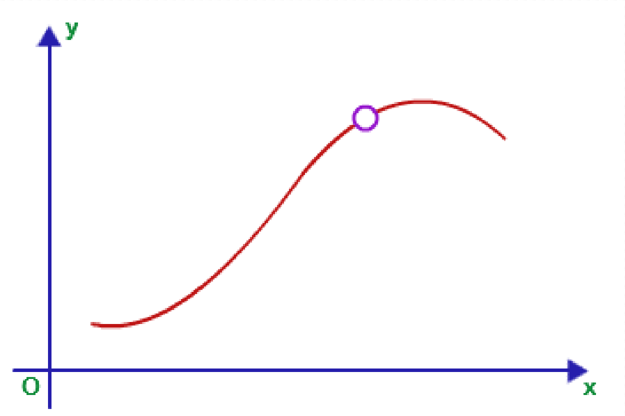
THIS IS THE DEFINITION OF CONTINUITY

Types of discontinuities:

1. **Removable Discontinuity:**

It is a type of discontinuity in which at a particular point, left hand side limit and right hand side limit are equal and finite, but are not equal to limit at that point. We can write that

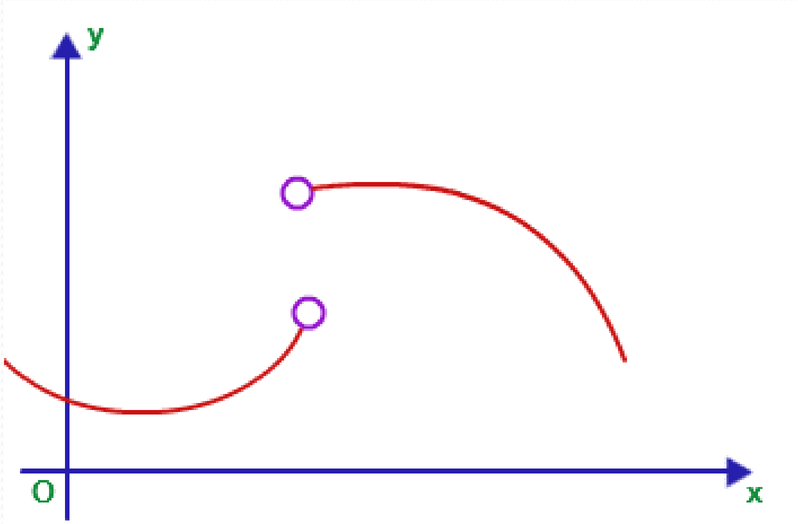
$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) \neq \lim_{x \to a} f(x)$$



Types of discontinuities:

2. **Jump Discontinuity:** In this discontinuity, left hand and right hand limits do exist and are finite, but are not equal. We can write as:

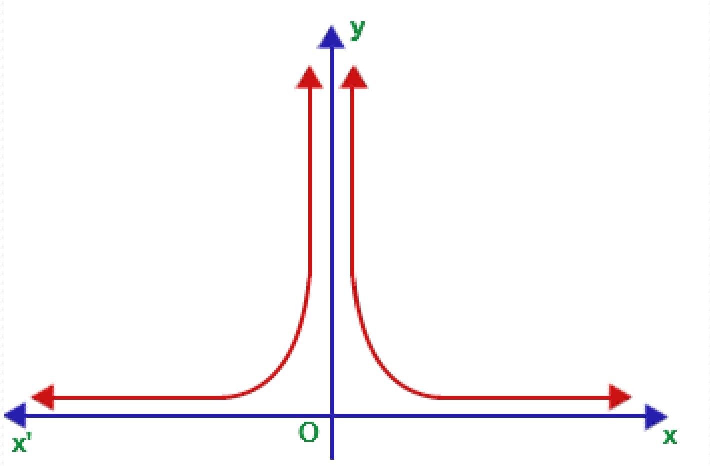
$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$



Types of discontinuities:

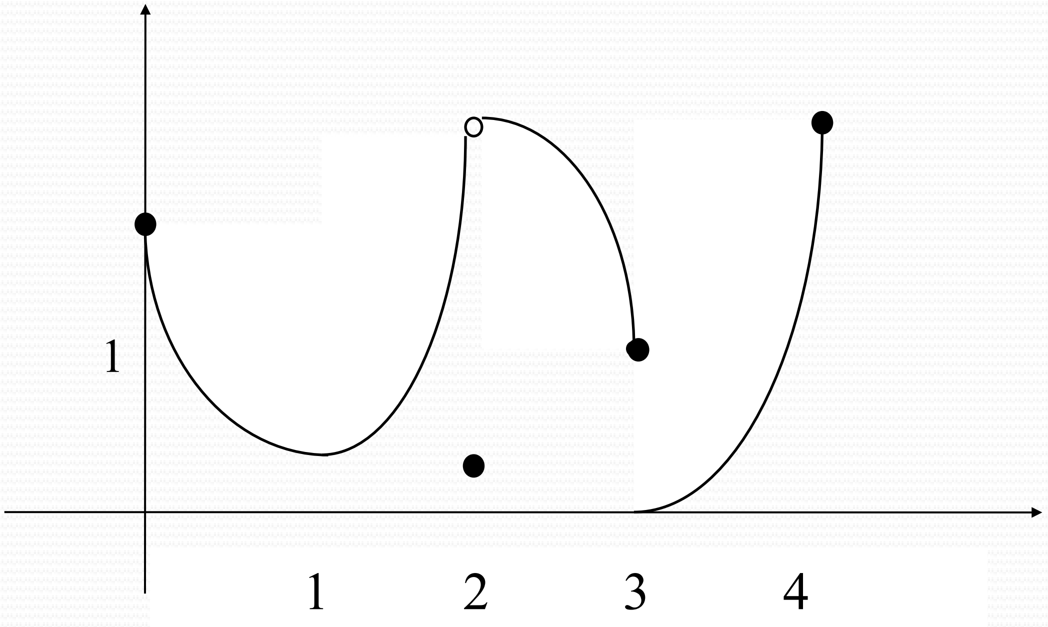
3. **Infinite Discontinuity:** Infinite discontinuity is a discontinuity in which one of left hand and right hand limits or both do not exist or are infinite. We can say that:

Either $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ or both do not exist or are infinite.



Example10:

Discuss the continuity at point x = 0, 2, 3, 3.5, 4



For , $x=0$

f is continuous at $x=0$ because

(a) $f(0)$ exists

(b) $\lim_{x\to 0} f(x) = 2$ exists

(c) $\lim_{x\to 0} f(x) = f(0)$

For , x=2

f is discontinuous at x =2 because

$$\lim_{x \to 2} f(x) \neq f(2)$$

The function fails **Part 3** of the test. The function also fails to be continuous from either right or left. The necessary limit exists but do not equal the function's value at x=2

For , x=3

f is discontinuous at x =3 because

$$\lim_{x \to 3} f(x)$$ Does not exists

The function fails **Part 2** of the test.
The function is continuous from **left** at x =3 because

$$\lim_{x \to 3^-} f(x) = f(3) = 1$$

But the function is not continuous from **right** at x =3 because

$$\lim_{x \to 3^+} f(x) = 0$$

While $f(3) = 1$

For , x=3.5

f is continuous at x =3.5 because

(a) $f(3.5)$ exists

(b) $\lim_{x \to 3.5} f(x)$ exists

(c) $\lim_{x \to 3.5} f(x) = f(3.5)$

For , x=4

f is continuous at x =4 because

(a) $f(4)$ exists

(b) $\lim_{x \to 4} f(x)$ exists

(c) $\lim_{x \to 4} f(x) = f(4)$

Week 4

Derivatives

Outlines:

- Differentiation of a Function
- Rules of Differentiation
- Higher order Differentiation
- Implicit Differentiation
- Application of Differentiation

Differentiation of a function

- The derivative is a fundamental calculus concept. To carefully understand the differentiation (derivative), the concept of a **limit** (Chapter 2) and the concept of a **tangent (slope)** must first be understood
- Keeping the above in mind, the *differentiation* of the function by using the first principle is:

$$f'(x) \equiv \frac{d}{dx}\left[f(x)\right] \equiv \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiation of a function

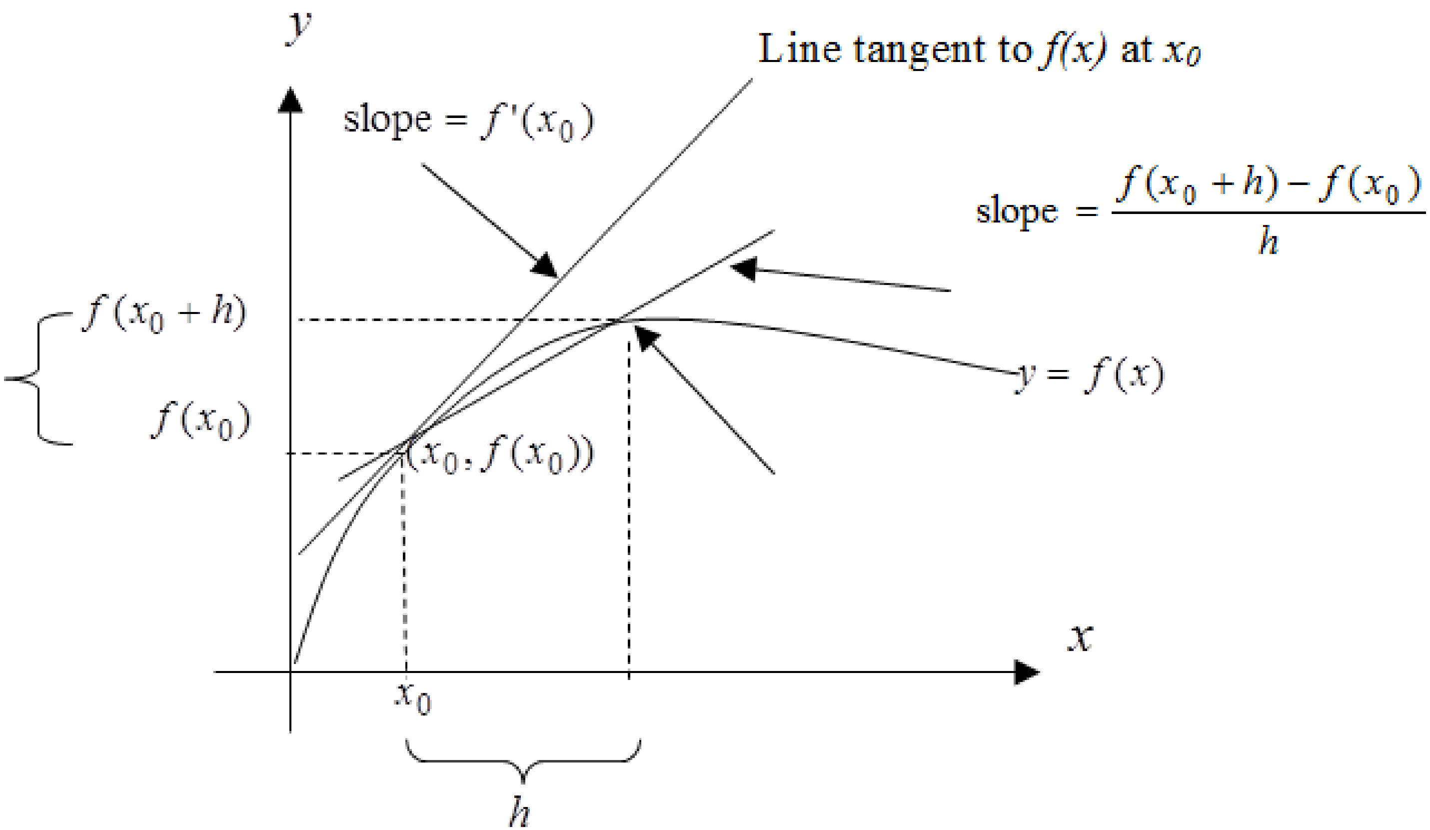


Figure 1: Defining the derivative geometrically

Differentiation of a function

- Intuitively, the derivative gives the change in the dependent variable that occurs when the independent variable increases by one unit, (where one unit is small).

- The idea behind the notion of differentiation is for example you leave Melaka by bus at 10 am on Saturday morning and arrive in Tapah at 2.15 pm. The distance between Melaka and Tapah is about 260 km, so the bus' average speed between Melaka and Tapah is

$$\frac{260}{(time\ of\ \ arrival)-(time\ of\ \ departure)}=\frac{260}{4.15}\approx 62.65\ km/h$$

Example 1:

- Find from the first principle, the derivative of y with respect to x if $y = x^2$

$$f'(x) \equiv \frac{d}{dx}\left[f(x)\right] \equiv \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh - h^2}{h}$$

$$= \lim_{h\to 0} 2x - h$$

$$= 2x$$

Derivative of a function at any point $x=a$

- Let a be a number in the domain of a function f. *If*

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

- exists, we call this limit the derivative of f at a and write

$$f'(a)$$

- So that

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

The expression $f'(a)$ is read 'the derivative of f at a' or 'f prime of a'
By definition, $f'(a)$ is the slope of the tangent line to the graph of f at $(a, f(a))$.

Example 2:

Find $f'(a)$ for the given value of *a =2*

$$f(x) = 3x^2 + 12$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$f'(2) = \lim_{x \to 2} \frac{3x^2 + 12 - (3(2^2) + 12)}{x - 2}$$

$$= \lim_{x \to 2} \frac{3x^2 - 12}{x - 2}$$

$$= \lim_{x \to 2} \frac{3(x-2)(x+2)}{x - 2}$$

$$= \lim_{x \to 2} 3(x+2)$$

$$=12$$

*RULES OF DIFFERENTIATION-1

Constant Rule	If $y = c$ then $\frac{dy}{dx} = \frac{d}{dx}\left[c\right] = 0$
Power Rule	If $y = x^c$ then $\frac{dy}{dx} = \frac{d}{dx}\left[x^c\right] = cx^{c-1}$
Constant Coefficient Rule	If $y = cf(x)$, then $\frac{dy}{dx} = \frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right] = cf'(x)$

Example 3:

1. If $y = 5$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}[5] = 0$

2. If $y = x^5$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}\left[x^5\right] = 5x^4$

3. If $y = 5x$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}[5x]$

$$= 5\frac{d}{dx}[x] = 5[1] = 5$$

*RULES OF DIFFERENTIATION-2

Sum Rule	If $y = f(x) + g(x)$ then $\frac{dy}{dx} = \frac{d}{dx}[f(x) + g(x)]$ $\frac{dy}{dx} = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$
Product Rule	If $y = f(x)g(x)$, then $\frac{dy}{dx} = \frac{d}{dx}[f(x)g(x)]$ $\frac{dy}{dx} = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$ $\frac{dy}{dx} = g(x)f'(x) + f(x)g'(x)$

Example 4: Sum Rule

If $y = 5x + 2x^2$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left[5x + 2x^2\right]$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[5x\right] + \frac{d}{dx}\left[2x^2\right]$$

$$\frac{dy}{dx} = 5 + 4x$$

Example 5: Product Rule

If $y = [5x][2x]$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left[[5x][2x]\right] = 2x\frac{d}{dx}[5x] + 5x\frac{d}{dx}[2x]$$

$$\frac{dy}{dx} = 2x[5] + 5x[2]$$

$$\frac{dy}{dx} = 10x + 10x = 20x$$

Don't

You can do it!!!

*RULES OF DIFFERENTIATION-4

Quotient rule	If $y = \frac{f(x)}{g(x}$, then $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$ $\frac{dy}{dx} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}$
Chain Rule	If y = f(g(x)), then $u = g(x)$ $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ $\frac{dy}{dx} = \frac{d}{dx}[f(g(x))]$ $= \frac{d}{d[g(x)]}[f(g(x))]\frac{d}{dx}[g(x)] = f'(g(x))g'(x)$

Example 6: Quotient Rule

If $y = \left(\dfrac{x^2 + 1}{x^3 + 2x} \right)$, then

$$\frac{d}{dx}\left(\frac{x^2 + 1}{x^3 + 2x} \right)$$

$$\frac{dy}{dx} = \frac{2x(x^3 + 2x) - (3x^2 + 2)(x^2 + 1)}{\left(x^3 + 2x\right)^2}$$

Example 7: Chain Rule

If $y = u^3 - 3u^2 + 1$ and $u = x^2 + 2$, then $\frac{dy}{dx}$

$$\frac{dy}{du} = 3u^2 - 6u \qquad \frac{du}{dx} = 2x$$

It follows from the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dx} . \frac{du}{dx}$$

$$\frac{dy}{dx} = (3u^2 - 6u)(2x)$$

Notice that this derivative expressed in terms of variables *x* and *u*.

To express $\frac{dy}{dx}$ in terms of x alone, we substitute $u = x^2 + 2$

$$\frac{dy}{dx} = \left(3(x^2+2)^2 - 6(x^2+2)\right)(2x) \quad = 6x^3(x^2+2)$$

*RULES OF DIFFERENTIATION-5

Power Rule	$y=[f(x)]^c$, then Let $u=f(x)$ and $y=[u]^c$ $\frac{dy}{du}=cu^{c-1}$ and $\frac{du}{dx}=f'(x)$ Using the Chain Rule $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$ $\frac{d[f(x)]^c}{dx}=c[f(x)]^{c-1}\cdot f'(x)$
Natural Log Rule	If $y=\ln(x)$, then $\frac{dy}{dx}=\frac{d}{dx}[\ln(x)]=\frac{1}{x}$

Example 8: Power Rule

If $y = (x+4)^3$, then $\frac{dy}{dx} = \frac{d}{dx}\left[(x+4)^3\right]$

Let $u = x + 4$ and $y = [u]^3$

$\frac{du}{dx} = 1$ $\frac{dy}{du} = 3u^2$

Using the Chain Rule $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

$$\frac{d}{dx}\left[(x+4)^3\right] = 3u^2 . 1$$

Substituting $u = x + 4$, $\frac{dy}{dx} = 3(x+4)^2$

Example 9: Natural Log (Ln) Rule

If $y = 5[\ln(x)]$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left[5\ln(x))\right]$$

$$\frac{dy}{dx} = 5\frac{d}{dx}\left[\ln(x))\right]$$

$$\frac{dy}{dx} = 5\left[\frac{1}{x}\right] = \frac{5}{x}$$

*RULES OF DIFFERENTIATION-6

Natural Exponential Rule	If $f(x) = e^x$, then $\frac{dy}{dx} = \frac{d}{dx}\left[e^x\right] = e^x$
Exponential Rule	If $f(x) = a^x$, then $\frac{dy}{dx} = \frac{d}{dx}\left[a^x\right] = a^x \ln a$ If $f(x) = a^u$, then $\frac{dy}{dx} = \frac{d}{dx}\left[a^u\right] = a^x (\ln a)\frac{du}{dx}$ Where $u = f(x)$

Example 10: Natural Exponential (e) Rule

If $y = e^{3x+4}$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left[e^{3x+4}\right]$$

Let $u = 3x + 4$ and $y = e^{u}$

Using the Chain Rule $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

$$\frac{dy}{dx} = e^{u} .3$$

$$\frac{dy}{dx} = 3e^{u} = 3e^{3x+4}$$

Example 11: Exponential Rule

1. If $y = 5^x$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left[5^x\right] = 5^x \ln 5$$

2. If $y = 4^{3x+5}$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left[4^u\right] = 4^u(\ln 4)\frac{du}{dx}$$ Where $u = 3x + 5$

$$\frac{dy}{dx} = 4^{3x+4}(\ln 4)3 = 3\left(4^{3x+4}(\ln 4)\right)$$

HIGHER ORDER DIFFERENTIATION

First derivative	y'	f'	$\frac{dy}{dx}$ or $\frac{d}{dx}[f(x)]$
Second derivative	y''	f''	$\frac{d}{dx}\left[\frac{dy}{dx}\right]=\frac{d^2y}{dx^2}$ or $\frac{d^2}{dx^2}[f(x)]$
Third derivative	y'''	f'''	$\frac{d}{dx}\left[\frac{d^2y}{dx^2}\right]=\frac{d^3y}{dx^3}$ or $\frac{d^3}{dx^3}[f(x)]$
Fourth derivative	$y^{(4)}$	$f^{(4)}$	$\frac{d}{dx}\left[\frac{d^3y}{dx^3}\right]=\frac{d^4y}{dx^4}$ or $\frac{d^4}{dx^4}[f(x)]$
⋮	⋮	⋮	⋮
n^{th} derivative	$y^{(n)}$	$f^{(n)}$	$\frac{d^n}{dx^n}[f(x)]$

Example 12:

Find $\dfrac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 7$.

$$2x^3 - 3y^2 = 7$$

$$6x^2 - 6y\,y' = 0$$

$$-6y\,y' = -6x^2$$

$$y' = \frac{-6x^2}{-6y}$$

$$y' = \frac{x^2}{y}$$

$$y'' = \frac{y \cdot 2x - x^2 y'}{y^2}$$

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}y'$$

Substitute y' back into the equation.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2} \cdot \frac{x^2}{y}$$

$$\boxed{y'' = \frac{2x}{y} - \frac{x^4}{y^3}}$$

Example 13:

Find the 1st, 2nd and 3rd derivatives of:

$f(x) = x^5 + 3x^4 - 2x^3 + 10x^2 + 5x - 12$

*IMPLICIT DIFFERENTIATION

How would you find the derivative in the equation

$$x^2 – 2y^3 + 4y = 2$$

where it is very difficult to express y as a function of *x*?

To do this, we use a procedure called **implicit differentiation.**

This means that when we differentiate terms involving *x* alone, we can differentiate as usual.

But when we differentiate terms involving *y*, we must apply the **Chain Rule**.

Example 14:

Differentiate the following with respect to *x*.

$3x^2$	$6x$
$2y^3$	$6y^2\ y'$
$x + 3y$	$1 + 3y'$
xy^2	$x(2y)y' + y^2(1) \quad = 2xyy' + y^2$

Product rule

Example 15:

Find *dy/dx* given that $y^3 + y^2 - 5y - x^2 = -4$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

Isolate dy/dx's

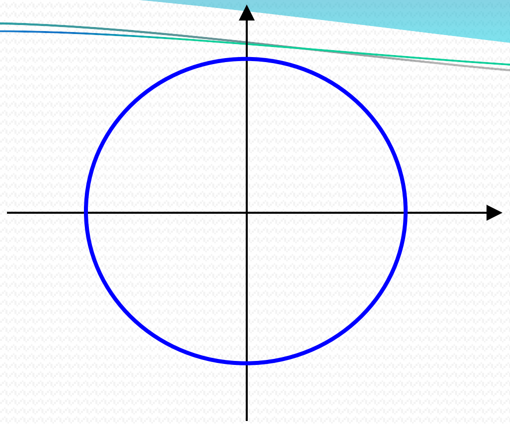
$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} = 2x$$

Factor out dy/dx

$$\frac{dy}{dx}\left(3y^2 + 2y - 5\right) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{\left(3y^2 + 2y - 5\right)}$$

Example 16:



This is not a function, but it would still be nice to be able to find the slope.

$$x^2 + y^2 = 1$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1$$

Do the same thing to both sides.

Note use of chain rule.

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Example 17:

Determine the slope of the tangent line to the graph of $x^2 + 4y^2 = 4$ at the point $\left(\sqrt{2}, -1/\sqrt{2}\right)$.

$$2x + 8y\frac{dy}{dx} = 0$$

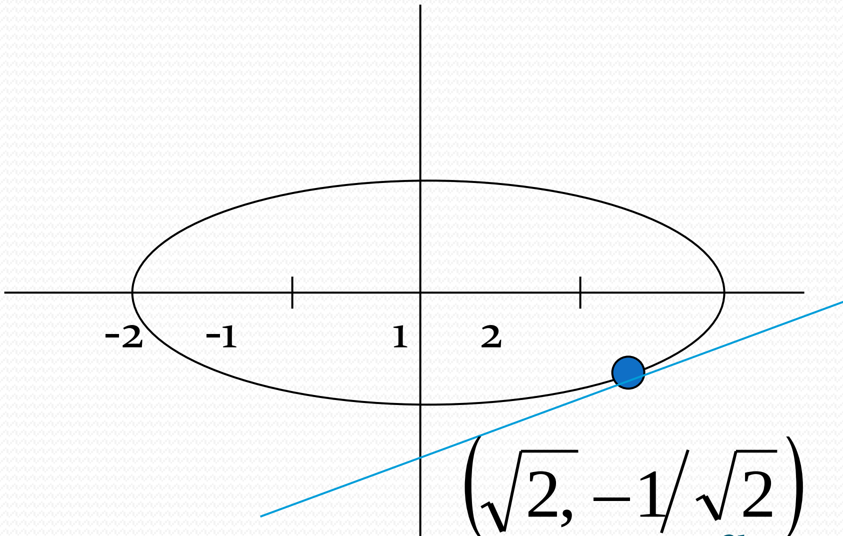
$$8y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{8y}$$

$$\frac{dy}{dx} = \frac{-x}{4y}$$

$$m = \frac{dy}{dx} = \frac{-\sqrt{2}}{4\left(\frac{-1}{\sqrt{2}}\right)} = \frac{-\sqrt{2}}{1}\cdot\frac{-\sqrt{2}}{4}$$

$$m = \frac{2}{4} = \frac{1}{2}$$



$\left(\sqrt{2}, -1/\sqrt{2}\right)$

APPLICATION OF DIFFERENTIATION

- Rectilinear Motion/ straight line motion
- Increasing and Decreasing Functions
- Stationary Points
- Second Order Derivatives
- Optimization Problems

Rectilinear Motion

If displacement from the origin is a function of time I.e. $x = f(t)$

then

$$v = \frac{dx}{dt}$$

v - velocity

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

a - acceleration

Example 18:

A body is moving in a straight line, so that after *t* seconds its displacement *x* metres from a fixed point O, is given by

$$x = 9t + 3t^2 - t^3$$

(a) Find the initial dislacement, velocity and acceleration of the body.

(b) Find the time at which the body is instantaneously at rest.

$$t = 0 \quad \Rightarrow x = 9(0) + 3(0)^2 - (0)^3 = 0 \text{ m}$$

$$v = \frac{dx}{dt} = 9 + 6t - 3t^2 \Rightarrow v = 9 \text{ m/s}$$

$$a = \frac{dv}{dt} = 6 - 6t \quad \Rightarrow \quad a = 6 \text{ m/s}^2$$

Increasing and decreasing functions

Look at how the value of the gradient changes as we move along this curve.

At any given point the function is either increasing, decreasing or stationary.

Increasing and decreasing functions

A function is said to be **increasing** when its gradient is positive.

So: A function $y = f(x)$ is increasing if $\frac{dy}{dx} > 0$.

A function is said to be **decreasing** when its gradient is negative.

So: A function $y = f(x)$ is decreasing if $\frac{dy}{dx} < 0$.

Is the function $f(x) = x^3 - 6x^2 + 2$ increasing or decreasing at the point where $x = 3$?

$$f'(x) = 3x^2 - 12x$$

$$f'(3) = 27 - 36 = -9$$

The gradient is negative, so the function is decreasing.

Example 19:

Suppose we want to know the range of values over which a function is increasing or decreasing.

Find the range of values of x for which the function $f(x) = x^3 - 6x^2 + 2$ is decreasing.

$$f'(x) = 3x^2 - 12x$$

$f(x)$ is decreasing when $f'(x) < 0$.

That is, when

$$3x^2 - 12x < 0$$

$$x^2 - 4x < 0$$

$$x(x - 4) < 0$$

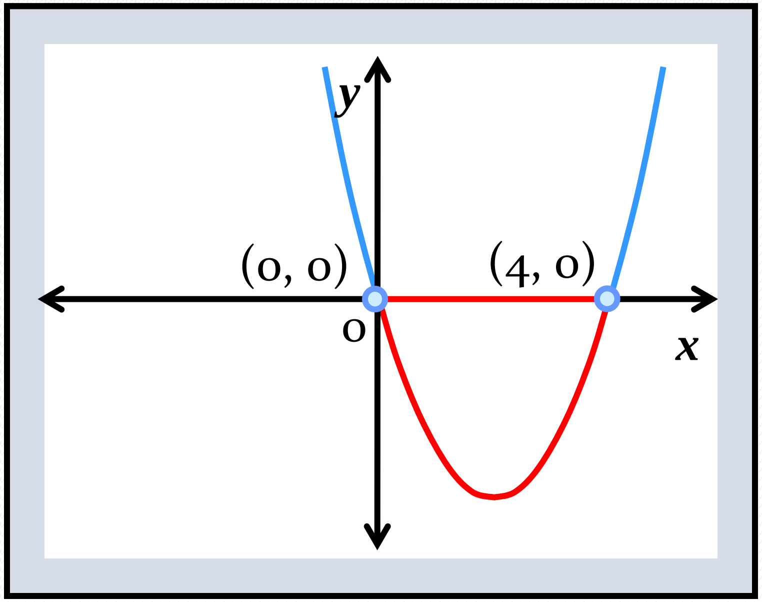
We can sketch the graph of $y = x(x - 4)$ to find the range for which this inequality is true.

Increasing and decreasing functions

The coefficient of $x^2 > 0$ and so the graph will be ∪-shaped.

Also, the roots of $y = x(x - 4)$ are $x = 0$ and $x = 4$.

This is enough information to sketch the graph.



The inequality

$$x(x - 4) < 0$$

is true for the parts of the curve that lie below the x-axis.

So $0 < x < 4$.

$f(x) = x^3 - 6x^2 + 2$ therefore decreases for $0 < x < 4$.

Stationary points

A **stationary point** occurs when the gradient of a curve is 0.

A stationary point can be:

- a **minimum** point
- a **maximum** point

Stationary points (Critical points)

Maximum and minimum stationary points are often called **turning points** because the curve turns as its gradient changes from positive to negative or from negative to positive.

A stationary point can also be a **point of inflection**.

Example 20:

We can find the coordinates of the stationary point on a given curve by solving $\frac{dy}{dx} = 0$.

Find the coordinates of the stationary points on the curve with equation $y = x^3 - 12x + 7$.

$$\frac{dy}{dx} = 3x^2 - 12$$

$\frac{dy}{dx} = 0$ when

$$3x^2 - 12 = 0$$

$$x^2 - 4 = 0$$

$$(x - 2)(x + 2) = 0$$

$$x = 2 \quad \text{or} \quad x = -2$$

Substituting $x = 2$ into $y = x^3 – 12x + 7$ gives

$$y = 2^3 – 12(2) + 7$$

$$= 8 – 24 + 7$$

$$= –9$$

So one of the stationary points has the coordinates (2, –9).

Substituting $x = –2$ into $y = x^3 – 12x + 7$ gives

$$y = (–2)^3 – 12(–2) + 7$$

$$= –8 + 24 + 7$$

$$= 23$$

So the other stationary point has the coordinates (–2, 23).

Deciding the nature of a stationary point

We can decide whether a stationary point is a maximum, a minimum or a point of inflection by working out whether the function is increasing or decreasing just before and just after the stationary point.

We have shown that the point (2, −9) is a stationary point on the curve $y = x^3 - 12x + 7$.

Let's see what happens when x is 1.9, 2 and 2.1.

Value of x	1.9	2	2.1
Value of $\frac{dy}{dx} = 3x^2 - 12$	−1.17	0	1.23
Slope	–ive	0	+ive

So (2, −9) is a **minimum** turning point.

Deciding the nature of a stationary point

Using this method for the other stationary point (–2, 23) on the curve $y = x^3 - 12x + 7$.

Value of x	-2.1	-2	-1.9
Value of $\frac{dy}{dx} = 3x^2 - 12$	1.23	0	−1.17
Slope	+ive	0	−ive

So (–2, 23) is a **maximum** turning point.

The main disadvantage of this method is that the behaviour of more unusual functions can change quite dramatically on either side of the turning point.

It is also time consuming and involves several calculations.

Using second order derivatives

Differentiating a function $y = f(x)$ gives us the derivative

$$\frac{dy}{dx} \quad \text{or} \quad f'(x)$$

Differentiating the function a second time gives us the **second order derivative**. This can be written as

$$\frac{d^2y}{dx^2} \quad \text{or} \quad f''(x)$$

The second order derivative gives us the rate of change of the gradient of a function.

We can think of it as the gradient of the gradient.

The second order derivative can often be used to decide whether a stationary point is a maximum point or a minimum point.

Using second order derivatives

If $\frac{d^2y}{dx^2} < 0$ at a stationary point, the point is a maximum.

If $\frac{d^2y}{dx^2} > 0$ at a stationary point, the point is a minimum.

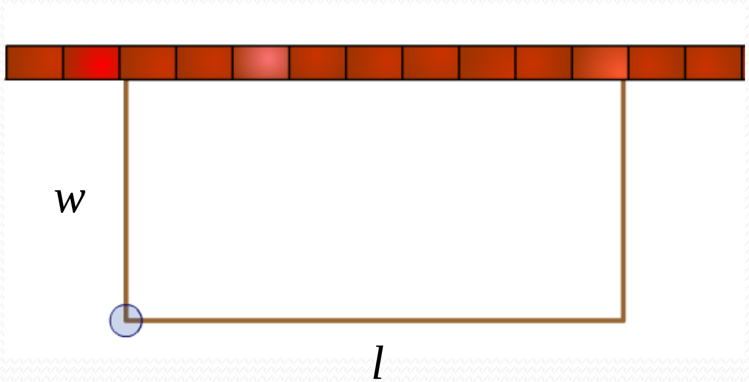
If $\frac{d^2y}{dx^2} = 0$ at a stationary point then the point could be a maximum, a minimum *or* a point of inflection.

In this case we would have to use the method of looking at the sign of the derivative at either side of the stationary point to decide its nature.

Optimization problems: Example 21

A very useful application of differentiation is in finding the solution to optimization problems.

A farmer has 60 m of fencing with which to construct a rectangular enclosure against an existing wall. Find the dimensions of the largest possible enclosure.



If l is the length of the enclosure and w is the width, we can write

$$2w + l = 60$$

$$l = 60 - 2w$$

The area A of the enclosure is:

$$A = wl$$

$$= w(60 - 2w)$$

$$= 60w - 2w^2$$

Look at how the value of *w* affects the area of the enclosure.

Example 22:

To find the maximum area we differentiate $A = 60w - 2w^2$ with respect to w.

$$\frac{dA}{dw} = 60 - 4w$$

$$\frac{dA}{dw} = 0 \text{ when } 60 - 4w = 0$$

$$4w = 60$$

$$w = 15$$

The maximum area is therefore achieved when the width of the enclosure is 15 m.

The dimensions of the enclosure in this case are 15 m by 30 m and the area is 450 m^2.

Example 23:

Given that $y = x^3 - 6x^2 - 15x$:

a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

b) Find the coordinates of any stationary points on the curve and determine their nature.

c) Sketch the curve.

a)

$$\frac{dy}{dx} = 3x^2 - 12x - 15$$

$$\frac{d^2y}{dx^2} = 6x - 12$$

b) The stationary points occur when $\frac{dy}{dx} = 0.$

$$3x^2 - 12x - 15 = 0$$

$$x^2 - 4x - 5 = 0$$

$$(x+1)(x-5) = 0$$

$x = -1$ or $x = 5$

When $x = -1$ $y = (-1)^3 - 6(-1)^2 - 15(-1)$

$= -1 - 6 + 15$

$= 8$

$$\frac{d^2y}{dx^2} = 6(-1) - 12$$

$$= -18$$

$\therefore$ $(-1, 8)$ is a maximum

When $x = 5$ $\quad y = (5)^3 - 6(5)^2 - 15(5)$

$= 125 - 150 - 75$

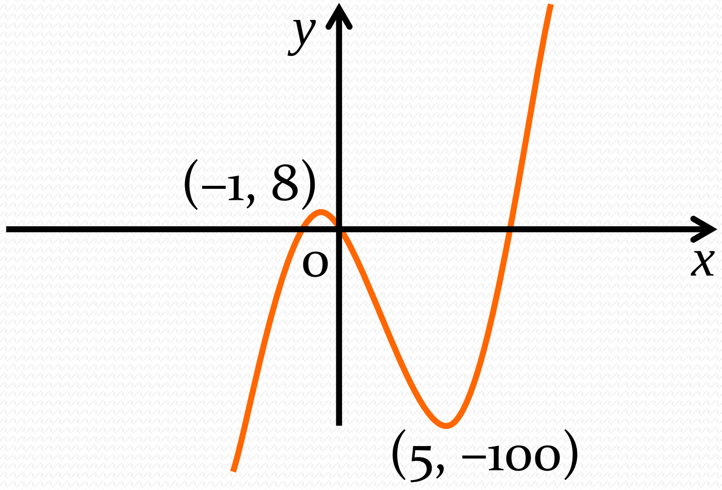
$= -100$

$$\frac{d^2y}{dx^2} = 6(5) - 12$$

$$= 18$$

$\therefore$ (5, –100) is a minimum

c) Sketching the curve:



Chapter 4

INTEGRATIONS

Outlines:

- **Indefinite Integration**
- **Definite Integration**
- **Improper Integration**
- **Techniques of Integration**
- **Application of Integration**

INTEGRATION

The reverse process to differentiation is called anti differentiation.
The process of finding anti derivatives is called integration.

How its done?

Differentiate x^2 → $2x^2$ → $2x^1$

Multiply by power   Drop power by 1

Integration x^2 ← $2x^2$ ← $2x^1$ Integration

Divide by new power   Raise power by 1

INDEFINITE INTEGRATION

The process of finding all antiderivatives of a function is called antidifferentiation, or integration.

$$\int f(x)dx$$

Integral sign **Integrand**

This is called the indefinite integral and is the most general antiderivative of f

We write

$$\int f(x)dx = F(x) + C$$

This is called an *indefinite integral*, f(x) is called the integrand and C is called the constant of integration.

BASIC INTEGRATION RULES

Rule 1: $\int k\,dx = kx + C$ (k, a constant)

Keep in mind that integration is the reverse of differentiation. What function has a derivative k?

kx + C, where C is any constant.

Another way to check the rule is to differentiate the result and see if it matches the integrand. Let's practice.

Example 1:

$$\int 2\,dx = 2x + C$$

Example 2:

$$\int \pi^2\,dx = \pi^2 x + C$$

Before we list Rule 2, let's go back and think about derivatives.

When we used the power rule to take the derivative of a power, we multiplied by the power and subtracted one from the exponent.

Example 3: $$\frac{d}{dx}\left(x^{3}\right)=3x^{2}$$

Since the opposite of multiplying is dividing and the opposite of subtracting is adding, to integrate we'd do the opposite. So, let's try adding 1 to the exponent and dividing by the new exponent.

Integrating: $$\int x^{3}dx=\frac{x^{3+1}}{3+1}=\frac{1}{4}x^{4}$$

Check by differentiating the result: $$\frac{d}{dx}\left(\frac{1}{4}x^{4}\right)=\frac{4}{4}x^{3}=x^{3}$$

Since we get the integrand we know it works.

BASIC INTEGRATION RULES

Rule 2: The Power Rule $\int x^n = \frac{x^{n+1}}{n+1} + C$ $n \neq -1$

Example 4: Find the indefinite integral $\int t^3 dt$

Solution: $\int t^3 dt = \frac{t^4}{4} + C$

Example 5: Find the indefinite integral $\int x^{\frac{3}{2}} dx$

Solution: $\int x^{\frac{3}{2}} dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C = \frac{X^{\frac{5}{2}}}{\frac{5}{2}} + C = \frac{2}{5} X^{\frac{5}{2}} + C$

Here are more examples of Rule 1 and Rule 2.

Example 6: Find the indefinite integral $\int \frac{1}{x^3}dx$

Solution: $\int \frac{1}{x^3}dx = \int x^{-3} = \frac{x^{-3+1}}{-3+1} + C = \frac{x^{-2}}{-2} + C = \frac{-1}{2x^2} + C$

Example 7: Find the indefinite integral $\int 1dx$

Solution: $\int 1dx = x + C$

Example 8: Find the indefinite integral $\int -3x^{-2}dx$

Solution: $\int -3x^{-2}dx = -3\int x^{-2} = \frac{-3x^{-2+1}}{-2+1} + C = \frac{-3x^{-1}}{-1} + C = \frac{3}{x} + C$

BASIC INTEGRATION RULES

Rule 3: The Indefinite Integral of a Constant Multiple of a Function

$$\int cf(x)dx = c\int f(x)dx, \text{ c is a constant}$$

Rule 4: The Sum Rule (or difference)

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$$

Rule 5: $\int e^x dx = e^x + C$

Rule 6: $\int \frac{1}{x}dx = \ln|x| + C$

To check these 2 rules, differentiate the result and you'll see that it matches the integrand.

Example 9: Integrate. $\int (2x - \frac{1}{x} + \frac{3}{x^2} + \sqrt{x} + 3e^x)dx$

Using the sum rule we separate this into 5 problems.

$$\int 2x dx - \int \frac{1}{x} dx + \int \frac{3}{x^2} dx + \int \sqrt{x} dx + \int 3e^x dx$$

Call them: 1 2 3 4 5

For ① we will use rule 3 to bring the constant outside of the integral sign.

$$2\int x dx$$

Next we will use rule 2, the power rule to integrate.

$$2\int x dx = 2\left(\frac{x^{1+1}}{1+1}\right) = 2\frac{x^2}{2} = x^2$$

Example 9 continues…

$$\int 2x\,dx - \int \frac{1}{x}dx + \int \frac{3}{x^2}dx + \int \sqrt{x}\,dx + \int 3e^x dx$$

Call them: 1 2 3 4 5

1 $\int 2x\,dx = x^2$

For 2 we will use Rule 6 the natural log rule.

$$-\int \frac{1}{x}dx = -\ln|x|$$

For 3 we will first rewrite then use the constant rule (Rule 3) and then the power rule (Rule 2).

$$\int \frac{3}{x^2}dx = 3\int x^{-2}dx = 3\frac{x^{-2+1}}{-2+1} = 3\frac{x^{-1}}{-1} = \frac{-3}{x}$$

Example 9 continues…

$$\int 2x\,dx - \int \frac{1}{x}dx + \int \frac{3}{x^2}dx + \int \sqrt{x}\,dx + \int 3e^x dx$$

Call them: 1 2 3 4 5

(1) $\int 2x\,dx = x^2$

(2) $-\int \frac{1}{x}dx = -\ln|x|$

(3) $\int \frac{3}{x^2}dx = \frac{-3}{x}$

For (4) we will rewrite and then use the power rule (Rule 2).

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3}x^{\frac{3}{2}}$$

For (5) we will use the constant rule (Rule 3) and then Rule 5 for e^x.

$$\int 3e^x dx = 3\int e^x dx = 3e^x$$

Example 9 continues…

$$\int 2x\,dx - \int \frac{1}{x}dx + \int \frac{3}{x^2}dx + \int \sqrt{x}\,dx + \int 3e^x dx$$

Call them: 1 2 3 4 5

$\int 2x\,dx$	$-\int \frac{1}{x}dx +$	$\int \frac{3}{x^2}dx$	$+\int \sqrt{x}\,dx$	$+\int 3e^x dx$
x^2	$-\ln\lvert x\rvert$	$+\frac{-3}{x}$	$+\frac{2}{3}x^{\frac{3}{2}}$	$+3e^x$

So in conclusion:

$$\int 2x\,dx - \int \frac{1}{x}dx + \int \frac{3}{x^2}dx + \int \sqrt{x}\,dx + \int 3e^x dx = x^2 - \ln\lvert x\rvert - \frac{3}{x} + \frac{2}{3}x^{\frac{3}{2}} + 3e^x + C$$

You may be wondering why we didn't use the C before now. Let's say that we had five constants $C_1 + C_2 + C_3 + C_4 + C_5$. Now we add all of them together and call them C. In essence that's what's going on above.

Review - Basic Integration Rules

Rule 1: $\int k\,dx = kx + C$ (k, a constant)

Rule 2: The Power Rule $\int x^n = \frac{x^{n+1}}{n+1} + C$

Rule 3: The Indefinite Integral of a Constant Multiple of a Function

$$\int cf(x)dx = c\int f(x)dx, \text{ c is a constant}$$

Rule 4: The Sum Rule (or difference)

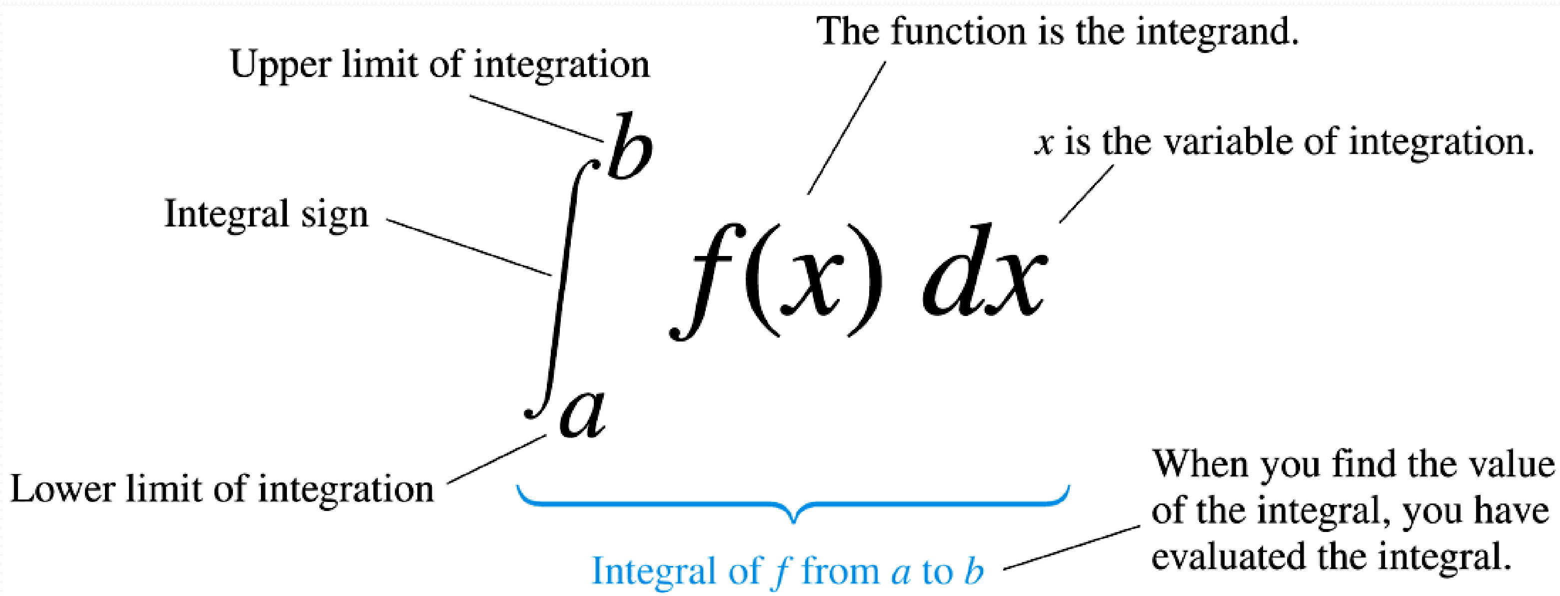
$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx$$

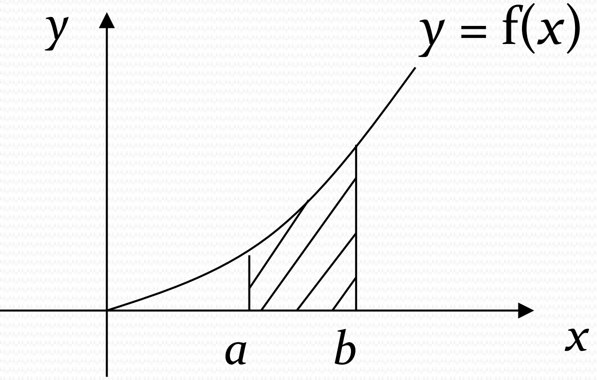
Rule 5: $\int e^x dx = e^x + C$

Rule 6: $\int \frac{1}{x}dx = \ln|x| + C$

DEFINITE INTEGRATION



The Definite Area



The notation for the area between the graph $y = \mathrm{f}(x)$ And the x-axis from $x = a$ to $x = b$ is

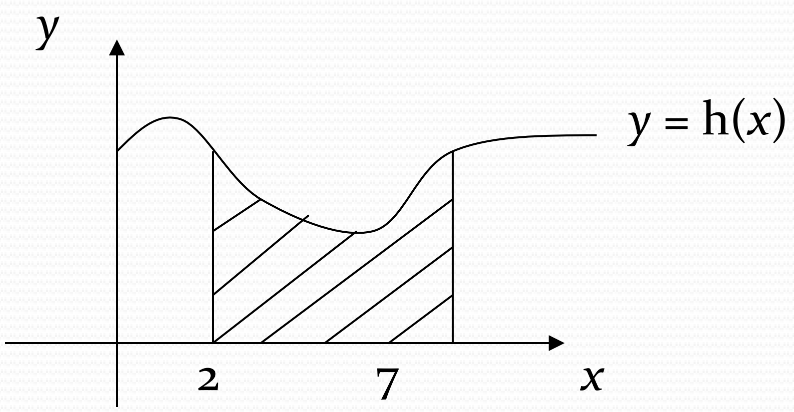
$$\int_a^b f(x)\,dx$$

This is called the definite integral. a and b are the lower and upper limits of integration, respectively.

The area between the graph of $y = \mathrm{f}(x)$ And the x-axis from $x = a$ to $x = b$ can be calculated as the area from $x = 0$ *to* $x = b$ minus the area from $x = 0$ *to* $x = a$.

Example 10:

Write the shaded area shown below as a definite integral.



$$\text{Area} = \int_2^7 h(x)\, dx$$

Example 11:

Find $\int_2^3 3x^2\, dx$

$$= [x^3]_2^3$$

$$= 3^3 - 2^3$$

$$= \underline{\underline{19\ units^2}}$$

Note: The constant of integration is cancelled out during the subtraction

As $c - c = 0$

Properties of Definite Integrations

P1 The value of the definite integral of a given function is a real number, depending on its lower and upper limits only, and is independent of the choice of the variable of integration,

$$\int_a^b f(x)\mathrm{dx} = \int_a^b f(y)\mathrm{dy} = \int_a^b f(t)\mathrm{dt}$$

P2

$$\int_a^b f(x)\mathrm{dx} = -\int_b^a f(x)\mathrm{dx}$$

P3

$$\int_a^a f(x)\mathrm{dx} = \int_b^b f(x)\mathrm{dx} = \int_a^b 0\,\mathrm{dx} = 0$$

P4

$$\int_a^b f(x)\mathrm{dx} = \int_a^c f(x)\mathrm{dx} + \int_c^b f(x)\mathrm{dx}\ , a \le c \le b$$

Properties of Definite Integrations

P6 Rules of Integration:

if f(x) , g(x) are continuous function on [a, b] then

$$\int_a^b kf(x)\mathrm{dx} = k\int_a^b f(x)\mathrm{dx}$$ for some constant k

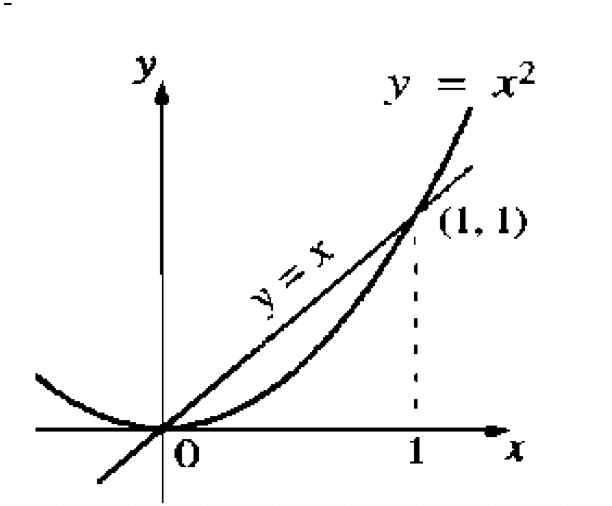
$$\int_a^b [f(x) \pm g(x)]\mathrm{dx} = \int_a^b f(x)\mathrm{dx} \pm \int_a^b g(x)\mathrm{dx}$$

Properties of Definite Integrations

P5 Comparison of two integrals, if,

$$f(x) \le g(x) \quad \forall x \in (a,b)$$

$$\int_a^b f(x)\mathrm{dx} \le \int_a^b g(x)\mathrm{dx}$$



$$x^2 \le x \quad , x \in (0,1)$$

$$\int_0^1 x^2\mathrm{dx} \le \int_0^1 x\,\mathrm{dx}$$

Properties of Definite Integrations

P7 $$\int_0^a f(x)\mathrm{dx} = \int_0^a f(a-x)\mathrm{dx}$$ a: any real constant.

$$\int_{-a}^{a} f(x)\,\mathrm{dx} = \int_0^a \left[f(x) + f(-x)\right]\mathrm{dx}$$

$$\int_0^{2a} f(x)\mathrm{dx} = \int_0^a \left[f(x) + f(2a-x)\right]\mathrm{dx}$$

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Example 12:

$$H(t) = \int_{-1}^{3} (t^3 + 1)\mathrm{dt}$$

$$H(t) = \left[\frac{\mathrm{t}^4}{4} + t \right]_{-1}^{3}$$

$$= \left[\frac{81}{4} + 3 \right] - \left[\frac{1}{4} - 1 \right]$$

$$= 24$$

USING THE PROPERTIES OF THE DEFINITE INTEGRAL

Example 13

Given: $\int_1^3 f(x)dx = 6$ $\int_3^7 f(x)dx = 9$ $\int_1^3 g(x)dx = -4$

$$\int_1^3 3f(x)dx = 3\int_1^3 f(x)dx = 3(6) = 18$$

$$\int_1^3 \left(2f(x) - 4g(x)\right)dx = 2\int_1^3 f(x)dx - 4\int_1^3 g(x)dx = 2(6) - 4(-4) = 28$$

$$\int_1^7 f(x)dx = \int_1^3 f(x)dx + \int_3^7 f(x)dx = 6 + 9 = 15$$

$$\int_3^1 f(x)dx = -\int_1^3 f(x)dx = -6$$

Example 14

$$\int_2^6 4\,dx = \quad 4(6-2) = \quad 16$$

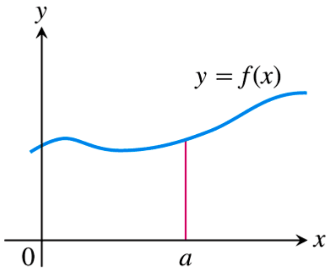
$$\int_4^8 x\,dx = \quad \frac{8^2}{2} - \frac{4^2}{2} = \quad 32 - 8 = \quad 24$$

$$\int_3^5 x^2\,dx = \frac{5^3}{3} - \frac{3^3}{3} = \quad \frac{125}{3} - \frac{27}{3} = \quad \frac{98}{3} = 32.67$$

$$\int_3^4 x^2 + 3x - 2\,dx = \int_3^4 x^2\,dx + 3\int_3^4 x\,dx - \int_3^4 2\,dx = \frac{4^3}{3} - \frac{3^3}{3} + 3\left(\frac{4^2}{2} - \frac{3^2}{2}\right) - \; 2(4-3) =$$

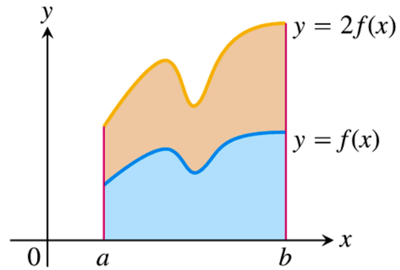
$$\frac{64}{3} - \frac{27}{3} + 3\left(\frac{16}{2} - \frac{9}{2}\right) - \; 2(1) = \quad 20.83$$

Geometric Interpretations of the Properties of the Definite Integral



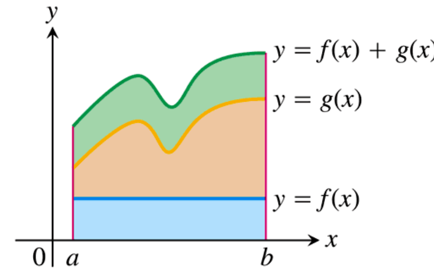
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



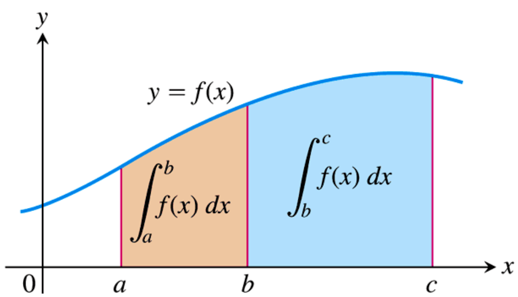
(b) *Constant Multiple:* ($k = 2$)

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



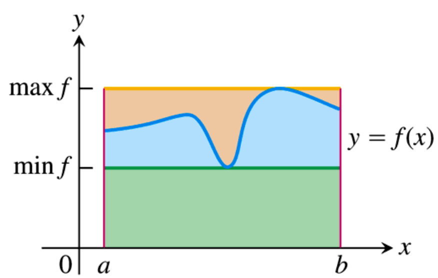
(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



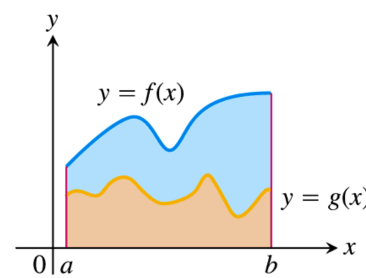
(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a)$$



(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

IMPROPER INTEGRATION

Definition A definite integration

$$\int_a^b f(x)\mathrm{dx}$$

is called **IMPROPER INTEGRATION** if the interval [a, b] of integration is infinite, or if $f(x)$ is not defined or not bounded at one or more points in [a, b]

$$\int_0^{+\infty} e^{-x}\,dx$$

IMPROPER INTEGRATION- definition

$$\int_a^{+\infty} f(x)\mathrm{dx}$$ **Defined** $$\lim_{b\to+\infty}\int_a^b f(x)\mathrm{dx}$$

$$\int_{-\infty}^{b} f(x)\mathrm{dx}$$ **Defined** $$\lim_{a\to-\infty}\int_a^b f(x)\mathrm{dx}$$

$$\int_{-\infty}^{+\infty} f(x)\mathrm{dx}$$ **Defined** $$\int_{-\infty}^{c} f(x)\mathrm{dx} + \int_c^{+\infty} f(x)\mathrm{dx}$$

$$\lim_{\ell\to-\infty}\int_\ell^c f(x)\mathrm{dx} + \lim_{\ell\to+\infty}\int_c^\ell f(x)\mathrm{dx}$$

The improper integration is said to be **Convergent** or **Divergent** according to the improper integration exists or not.

Example 15:

Evaluate $\int_1^{\infty} \frac{1}{x^2} dx$ **Defined** $\lim_{t \to +\infty} \int_1^t \frac{1}{x^2} \text{dx}$

$$\int_1^t \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^t = 1 - \frac{1}{t}$$

$$\int_1^t \frac{1}{x^2} dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^2} dx$$

$$= \lim_{t \to \infty} \left(1 - \frac{1}{t}\right)$$

$$= 1$$

Example 16:

Determine if the following integral is convergent or divergent and if it's convergent find its value.

$$\int_{1}^{\infty} \frac{1}{x}\, dx$$

$$\int_{1}^{\infty} \frac{1}{x}\, dx = \lim_{t\to\infty} \int_{1}^{t} \frac{1}{x}\, dx$$

$$= \left[\lim_{t\to\infty} \ln(x)\right]_{1}^{t}$$

$$= \lim_{t\to\infty} (\ln(t) - \ln(1))$$

$$= \infty$$

So, the limit is infinite and so the integral is divergent.

Techniques of Integrations

Integration by Substitution

Integration by Part

Integration by Substitution

The chain rule allows us to differentiate a wide variety of functions, but we are able to find anti-derivatives for only a limited range of functions?

We can sometimes use substitution or change of variable to rewrite functions in a form that we can integrate.

Steps for Integrating by Substitution

1. Choose a substitution $u = g(x)$, such as the inner part of a composite function.
2. Compute $du = g'(x)dx$
3. Re-write the integral in terms of u and du.
4. Find the resulting integral in terms of u.
5. Substitute $g(x)$ back in for u, yielding a function in terms of x only.
6. Check by differentiating.

Example 17:

$$\int (x+2)^5 \, dx$$

Let $u = x + 2$

$$du = dx$$

$$= \int u^5 du$$

$$= \frac{1}{6} u^6 + c$$

$$= \frac{(x+6)^6}{6} + c$$

<u>Don't</u> forget to substitute the value for *u* back into the problem!

→

Example 18:

$$\int \sqrt{1+x^2} \cdot 2x\, dx$$

One of the clues that we look for is if we can **find a function** and **its derivative** in the integral.

The derivative of $1+x^2$ is $2x\, dx$

$$\text{Let } u = 1+x^2$$

$$du = 2x\, dx$$

$$\int u^{\frac{1}{2}}\, du$$

$$\frac{2}{3}u^{\frac{3}{2}} + C$$

$$\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + C$$

Note that this only worked because of the 2*x* in the original.
Many integrals can not be done by substitution.

→

Example 19:

$$\int \sqrt{4x-1}\,dx$$

Let $u = 4x-1$

$$du = 4\,dx$$

$$\frac{1}{4}du = dx$$

Solve for *dx*.

$$\int u^{\frac{1}{2}} \cdot \frac{1}{4}du$$

$$\frac{2}{3}u^{\frac{3}{2}} \cdot \frac{1}{4} + C$$

$$\frac{1}{6}u^{\frac{3}{2}} + C$$

$$\frac{1}{6}(4x-1)^{\frac{3}{2}} + C$$

→

Example 20:

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\ dx$$

Let $u = x^3 + 1$ $\quad u(-1) = 0$

$du = 3x^2\ dx$ $\quad u(1) = 2$

$$\int_0^2 u^{\frac{1}{2}}\ du$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^2$$

Don't forget to use the new limits.

$$= \frac{2}{3}\cdot 2^{\frac{3}{2}} \quad = \frac{2}{3}\cdot 2\sqrt{2} \quad = \frac{4\sqrt{2}}{3}$$

→

Integration by Parts

If u and v are functions of x and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du$$

Proof:

Product Rule: $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\int \frac{d}{dx}(uv)dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$uv = \int udv + \int vdu$$

$$\int udv = uv - \int vdu$$

Summary of Common Integrals Using Integration by Parts

1. **For integrals of the form**

$$\int x^n e^{ax} dx, \quad \int x^n \sin ax dx, \quad \int x^n \cos ax dx$$

Let u = x^n and let dv = e^{ax} dx, sin ax dx, cos ax dx

2. **For integrals of the form**

$$\int x^n \ln x dx, \quad \int x^n \arcsin ax dx, \quad \int x^n \arccos ax dx$$

Let u = lnx, arcsin ax, or arctan ax and let dv = x^n dx

3. **For integrals of the form**

$$\int e^{ax} \sin bx dx, \quad \textbf{or} \quad \int e^{ax} \cos bx dx,$$

Let u = sin bx or cos bx and let dv = e^{ax} dx

Guidelines for Integration by Parts

1. Try letting dv be the most complicated portion of the integrand that fits a basic integration formula. Then u will be the remaining factor(s) of the integrand.

2. Try letting u be the portion of the integrand whose derivative is a simpler function than u. Then dv will be the remaining factor(s) of the integrand.

TIPS

Pick your "u" and "dv."

TRICK: **L.I.P.E.T.** This determines what u should be initially.

L: Logarithm,
I : Inverse,
P: Power,
E: Exponential
T: Trig functions.

$$\int xe^x \, dx$$

In this problem, a power function and an exponential function are present.
Since power (P) comes before exponential (E), u should equal x.
From the u, find du. From the dv, find v.
Therefore u=x, then du=dx! Don't forget that.
Simply plug this into the "parts formula."

Example 21:

Evaluate

$$\int xe^x \, dx$$

To apply integration by parts, we want to write the integral in the form $\int u \, dv.$

There are several ways to do this.

$$\underbrace{\int (x)}_{u}\underbrace{(e^x dx)}_{dv} \qquad \underbrace{\int (e^x)}_{u}\underbrace{(x dx)}_{dv} \qquad \underbrace{\int (1)}_{u}\underbrace{(xe^x dx)}_{dv} \qquad \underbrace{\int (xe^x)}_{u}\underbrace{(dx)}_{dv}$$

Following our guidelines, we choose the first option because the derivative of $u = x$ is the simplest and $dv = e^x \, dx$ is the most complicated.

$$\int (x)(e^x dx)$$

$\underbrace{x}_{u}$ $\underbrace{e^x dx}_{dv}$

u = x

v = e^x

du = dx

dv = e^x dx

$$\int u\,dv = uv - \int v\,du$$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

Example 22:

$$\int x^2 \ln x \; dx$$

Since x^2 integrates easier than $\ln x$, let $u = \ln x$ and dv = x^2

u = ln x

dv = x² dx

$$du = \frac{1}{x} dx$$

$$v = \frac{x^3}{3}$$

$$\int u\,dv = uv - \int v\,du$$

$$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{1}{x}dx \quad = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}dx$$

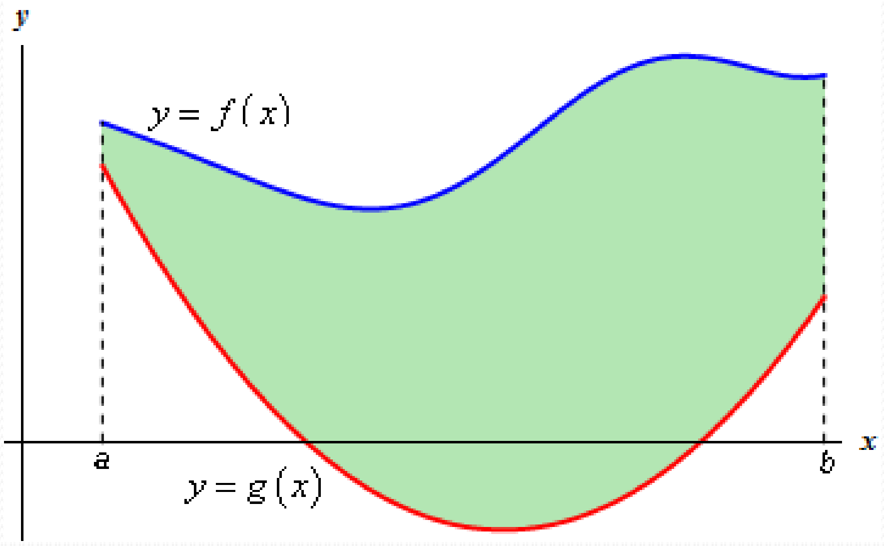
$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

Application of Integration

Area between Curves

Volumes of Solids of Revolution

Area between Curves- Case 1

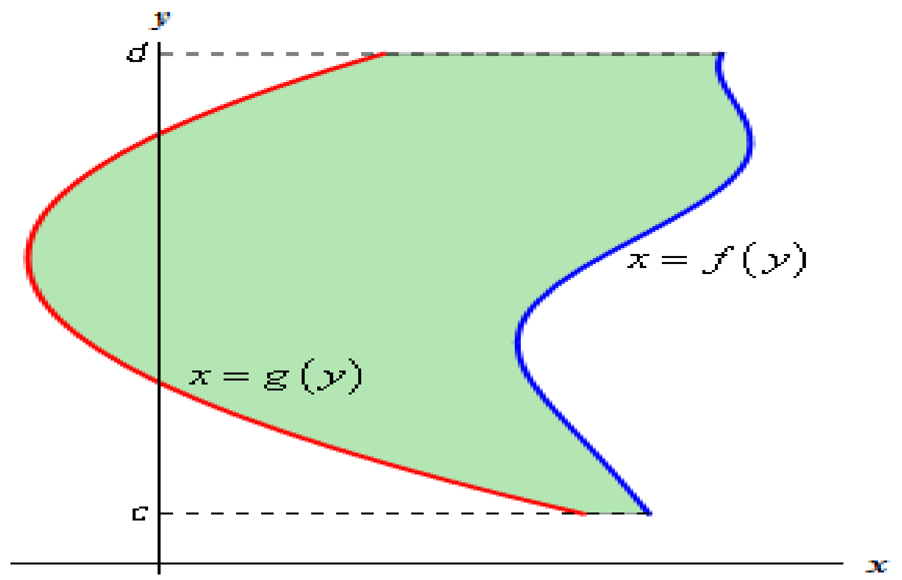


Therefore the area between these two curves is

$$A = \int_a^b f\,(x) - g(x)$$

$$A = \int_a^b \binom{\text{Upper}}{\text{function}} - \binom{\text{Lower}}{\text{function}}\, dx \quad a \leq x \leq b$$

Area between Curves- Case 2



The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[\, c, d\,]$ with $f(y) \geq g(y)$

In this case the formula is,

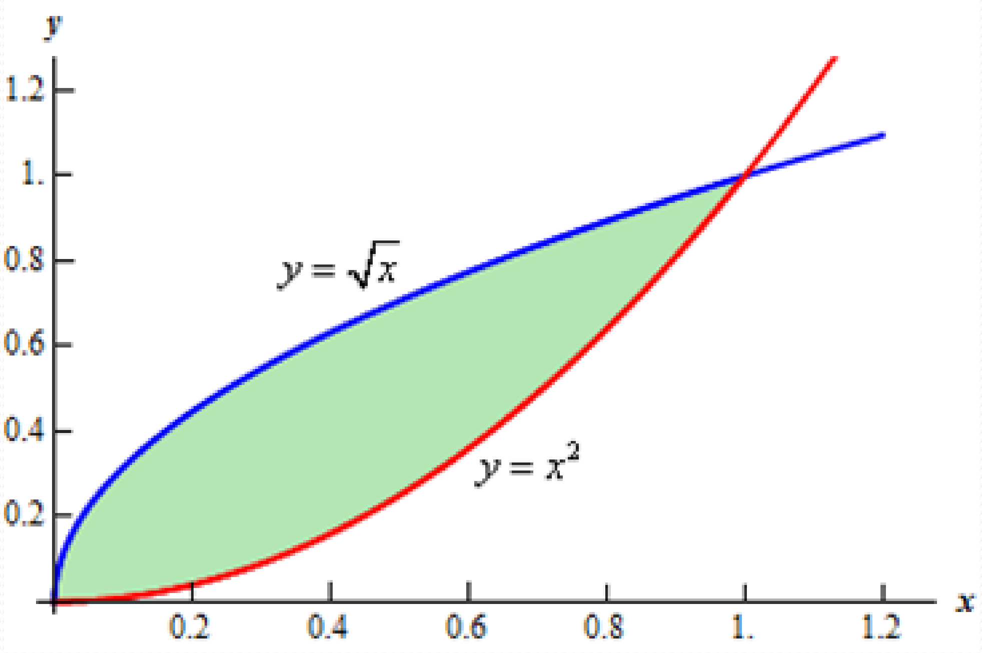
$$A = \int_c^d f(y) - g(y)$$

$$A = \int_c^d \begin{pmatrix} \text{Right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{Left} \\ \text{function} \end{pmatrix} dx \quad c \leq x \leq d$$

Example 23

Determine the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$

STEP 1: Sketch the curves. Identifying the upper and the lower curves, we take $f(x) = x^2$ and $f(x) = \sqrt{x}$.



STEP 2: The x- coordinates of the intersection point are the limits of the integration. Find the limit by solving $y = x^2$ and y= $\sqrt{x}$ simultaneously for

$$x^2 = \sqrt{x}$$

$$\therefore \text{x} = 0 \text{ and x} = 1;$$

STEP 3: Integrate

$$A = \int_a^b \begin{pmatrix}\text{Upper}\\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{Lower}\\ \text{function}\end{pmatrix} dx$$

$$A = \int_0^1 (\sqrt{\text{x}}) - (\text{x}^2) dx$$

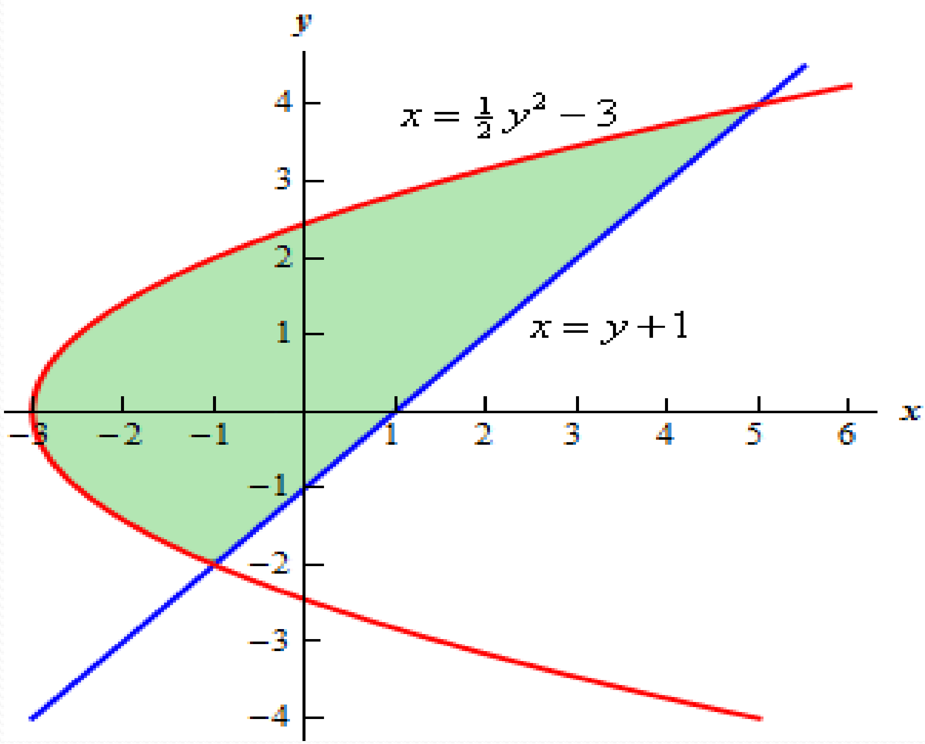
$$A = \left[\frac{2}{3}\sqrt[3]{x} - \frac{1}{3}\ x^3\right]_0^1$$

$$A = \frac{1}{3}$$

Example 24

Determine the area of the region enclosed by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$

STEP 1: Sketch the curves. Identifying the upper and the lower curves, we take $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$



STEP 2: The x- coordinates of the intersection point are the limits of the integration. Find the limit by solving $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$ simultaneously for

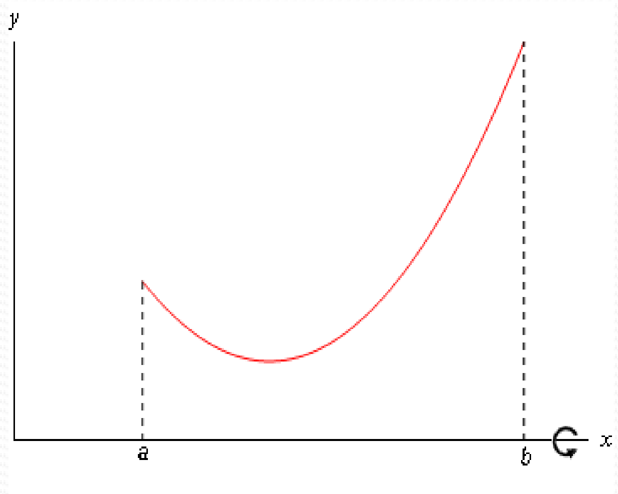
$$y + 1 = \frac{1}{2}y^2 - 3$$
$$2y + 2 = y^2 - 6$$
$$0 = y^2 - 2y - 8$$
$$0 = (y - 4)(y + 2)$$

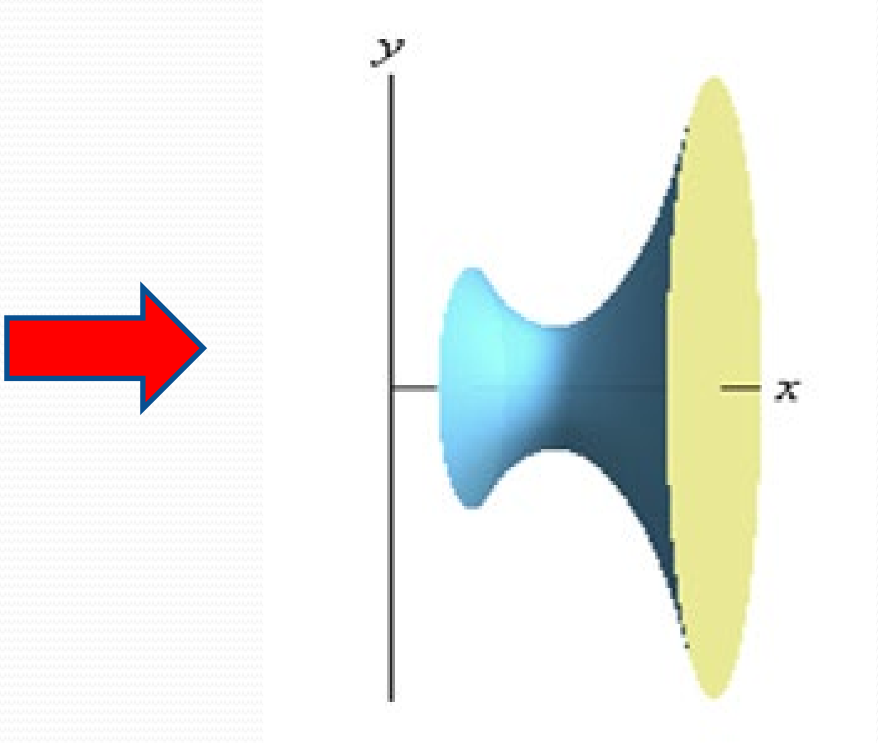
STEP 3: Integrate

$$A = \int_a^b \binom{\text{Right}}{\text{function}} - \binom{\text{Left}}{\text{function}} dy$$
$$A = \int_{-2}^{4} (y + 1) - \left(\frac{1}{2}y^2 - 3\right) dy$$
$$A = \int_{-2}^{4} \left(\frac{1}{2}y^2 + y + 4\right) dy = \left[-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right]_{-2}^{4}$$
$$A = 18$$

Volumes of Solids of Revolution

In this section we will start looking at the volume of a solid of revolution. We should first define just what a solid of revolution is. To get a solid of revolution we start out with a function, $y=f(x)$ on an interval $[a, b]$.





Volumes of Solids of Revolution

The volume of the solid generated by revolving the region about the x - axis between the graph of the continuous function $y = f(x)$ and from $x = a$ to $x = b$ is:

$$V = \int_a^b \pi\,(radius\ function)^2\,dx$$

$$V = \int_a^b \pi\,(y)^2\,dx$$

$$V = \int_a^b \pi\,(f(x))^2\,dx$$

Volumes of Solids of Revolution

The volume of the solid generated by revolving the region about the y - axis between the graph of the continuous function $x = g(y)$ and from $y = a$ to $y = b$ is:

$$V = \int_a^b \pi \, (radius\ function)^2 \, dy$$

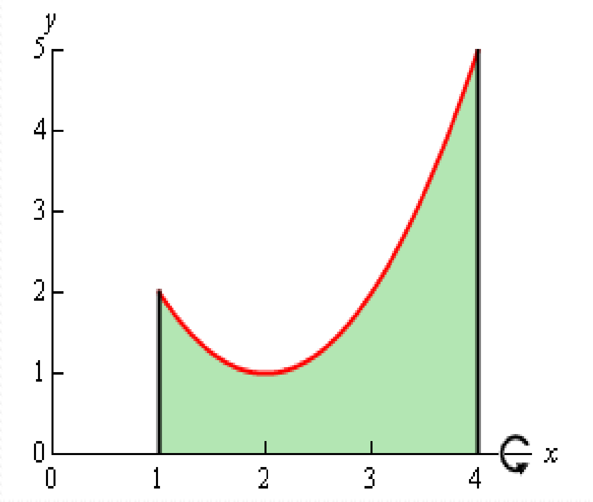
$$V = \int_a^b \pi \, (x)^2 \, dy$$

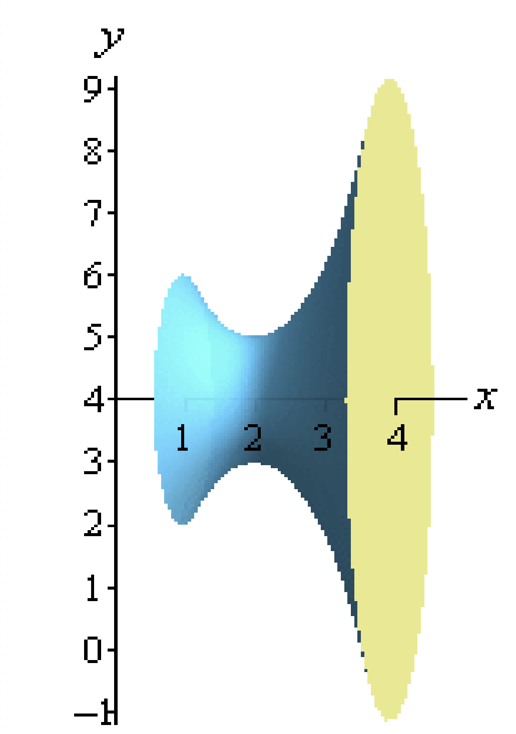
$$V = \int_a^b \pi \, (g(y))^2 \, dy$$

Example 25

Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$ and $x = 1$, $x = 4$ about the x–axis

STEP 1: Sketch the region





STEP 2: The volume of this solid is then

$$V = \int_a^b \pi\, (f(x))^2\, dx$$

$$V = \int_a^b \pi\, (x^2 - 4x + 5)^2\, dx$$

$$V = \int_1^4 \pi\, (x^4 - 8x^3 + 26x^2 - 40x + 25)\, dx$$

$$V = \left[\pi \left(\frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x\right)\right]_1^4$$

$$V = \frac{78\pi}{5}$$

Chapter 5

INTRODUCTION TO DIFFERENTIAL EQUATIONS

Outlines

- First-order differential equation
- Application of Differential Equation

Introduction

- One of the most important application of calculus is differential equations, which often arise in describing some phenomenon in engineering, physical science and social science as well.
- In general, a **differential equation** is an equation that contains an unknown function and its derivatives. The **order** of a differential equation is the order of the highest derivative that occurs in the equation.
- A function y=f(x) is called a **solution** of a differential equation if the equation is satisfied when y=f(x) and its derivatives are substituted into the equation.

Introduction

- If no additional conditions, the solution of a differential equation always contains some constants. The solution family that contains arbitrary constants is called the **general solution.**
- In real applications, some additional conditions are imposed to uniquely determine the solution. The conditions are often taken the form that is, giving the value of the unknown function at the end point. This kind of condition is called an **initial condition**, and the problem of finding a solution that satisfies the initial condition is called an **initial-value problem**.

Introduction

- A differential equation is an equation connecting a function and its derivatives.
- A first-order differential equation involves only first derivatives. Examples of first-order differential equations are

$$\frac{dy}{dx} + 2xy = 3x^2$$

$$(3x-2)\frac{dy}{dx} = y^2$$

$$\frac{dy}{dx} + 3y = 2x - 1$$

Introduction

- A second-order differential equation involves second derivatives. Examples of second-order differential equations are

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{4x}$$

$$\frac{d^2y}{dx^2} + 4y = \sin x$$

First-order differential equation

1. Finding the complementary function and a particular solution
2. Separating the variables
3. Integrating Factors

Finding the complementary function and a particular solution

- This method is suitable for solving only linear first-order differential equations. Suppose that the differential equation has the form

$$a\frac{dy}{dx}+by=f(x)$$

Finding the complementary function and a particular solution

<u>Step 1:</u> Find the **complementary function**, that is the solution to the equation $a\dfrac{dy}{dx} + by = 0$.

The complementary function is $y = Ae^{mx}$

where m is the solution to the **auxiliary equation**, $am + b = 0$

<u>Step 2:</u> Find the **particular solution** that is a function that satisfies the original differential equation.

<u>Step 3:</u> The general solution to the original differential equation is
y = complementary function + particular solution

Example 1:

Find the general solution to the differential equation

$$2\frac{dy}{dx} - 3y = e^{2x}$$

Solution

STEP 1:

The auxiliary equation is $2m - 3 = 0$.

$$\Rightarrow m = 1.5$$

Therefore the complimentary function is

$$y = Ae^{1.5x}$$

Example 1:

Solution

STEP 2:

The **particular solution** is $y = ae^{2x}$

$$\Rightarrow \frac{dy}{dx} = 2ae^{2x}$$

Substituting these into the original differential equation gives: $2\frac{dy}{dx} - 3y = e^{2x}$

$$2 \times 2ae^{2x} - 3ae^{2x} = e^{2x}$$

$$ae^{2x} = e^{2x}$$

$$a = 1$$

Therefore the particular solution is

$$y = e^{2x}$$

Example 1:

Solution

STEP 3:

The general solution to the original differential equation is

$$y = Ae^{1.5x} + e^{2x}$$

Separating The Variables

This method only works for differential equations which can be rearranged to the form

$$f(x)\frac{dy}{dx} = g(y)$$

Suppose that the differential equation has the form

$$g(y)\frac{dy}{dx} = f(x)$$

The variables can then be separated out:

$$\int g(y)dy = \int f(x)dx$$

Separating The Variables

Step 1: Separate the variables y from x, i.e., by collecting on one side all terms involving y together with dy, while all terms involving x together with dx are put on the other side.

$$g(y)\frac{dy}{dx} = f(x)$$

Step 2: Integrate both sides.

$$\int g(y)dy = \int f(x)dx$$

Example 2:

Find the solution to the differential equation

$$x^2 \frac{dy}{dx} = (x+1)(y+1)$$

given that $y = 2e$ when $x = 1$.

Solution

STEP 1:

separate out the variables

$$x^2 dy = (x+1)(y+1)dx$$

$$\frac{1}{y+1} dy = \frac{x+1}{x^2} dx$$

Example 2:

Solution

STEP 2:

Integrate both sides

$$\int \frac{1}{y+1} dy = \int \frac{x+1}{x^2} dx$$

Evaluating the right hand side:

$$\int \frac{x+1}{x^2} dx = \int \left(\frac{x}{x^2} + \frac{1}{x^2} \right) dx = \int \left(\frac{1}{x} + x^{-2} \right) dx$$

$$= \ln x - x^{-1} + c$$

Evaluating the right hand side:

$$\int \frac{1}{y+1} dy = \ln(y+1)$$

Example 2:

Solution

STEP 2:

Integrate both sides

$$\int \frac{1}{y+1} dy = \int \frac{x+1}{x^2} dx$$

Put together:

$$\ln(y+1) = \ln x - x^{-1} + c$$

Take exponentials of both sides:

$$y+1 = e^{\ln x - x^{-1} + c}$$

$$y+1 = e^{\ln x} e^{x^{-1}} e^{c}$$

$$y = Axe^{x^{-1}}$$

Example 2:

Solution

Substitute in $y = 2e$ and $x = 1$:

$$2e = Ae^{1}$$

Therefore $A = 2$

So the solution is $y = 2xe^{x^{-1}}$

Integrating Factors

This method is used when the equation is in the form of linear equation in which the variables cannot be separated

$$\frac{dy}{dx} + p(x)y = q(x)$$ (Basic form)

where $p(x)$ and $q(x)$ – continuous functions, may or may not be constants

Some other examples

a) $\frac{dy}{dx} + x^2 y = e^x \Rightarrow p(x) = x^2; \quad q(x) = e^x$

b) $\frac{dy}{dx} + (\sin x)y + x^3 = 0 \Rightarrow p(x) = \sin x; \quad q(x) = -x^3$

c) $\frac{dy}{dx} + 5y = 2 \Rightarrow p(x) = 5; \quad q(x) = 2$

Integrating Factors

Step 1: Must remember the basic form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

Step 2: Multiply both sides by μ gives

$$\mu\frac{dy}{dx} + \mu p(x)y = \mu q(x)$$

$\mu = e^{\int p(x)dx}$ - is called the integrating factor.

Step 3: Integrate both sides of the equation obtained in (2) and the result obtained is :

$$\mu y = \int \mu q(x)\,dx$$

Example 3:

Solve the following differential equations :

$$\frac{dy}{dx} + 3y = e^{-3x}$$

$$Compared \quad with \quad the \quad basic \quad form, \quad p(x) = 3$$

$$so \quad \mu = e^{\int p(x)dx} = e^{\int 3dx} = e^{3x}$$

$$\therefore \frac{dy}{dx} + 3y = e^{-3x} \qquad \Rightarrow \mu y = \int \mu q(x)\, dx$$

$$e^{3x} y = \int e^{3x} e^{-3x}\, dx$$

$$e^{3x} y = \int dx$$

$$e^{3x} y = x + C$$

$$\therefore \quad y = \frac{x + C}{e^{3x}}$$

Application of Differential Equation

Exponential Growth and Decay

- In this section, we examine how population growth can be modeled using differential equations. We start with the basic exponential growth and decay models.

$$\frac{dP}{dt} = kP$$

Basic background ideas from algebra and calculus

1. A variable y is *proportional* to a variable x if $y = k\,x$, where k is a constant.
2. Given a function $P(t)$, where P is a function of the time t, the rate of change of P with respect to the time t is given by $\frac{dP}{dt} = P'(t)$.
3. A function $P(t)$ is increasing over an interval if $\frac{dP}{dt} = P'(t) > 0$.

   A function $P(t)$ is decreasing over an interval if $\frac{dP}{dt} = P'(t) < 0$.

   A function $P(t)$ is neither increasing or decreasing over an interval if $\frac{dP}{dt} = P'(t) = 0$.

Exponential Growth and Decay

When a population grows exponentially, it grows at a rate that is proportional to its size at any time t. Suppose the variable $P(t)$ (sometimes we use just use P) represents the population at any time t. In addition, let P_0 be the initial population at time $t = 0$, that is, $P(0) = P_0$

Then if the population grows exponentially,

(Rate of change of population at time t) = k (Current population at time t)

In mathematical terms, this can be written as

$$\frac{dP}{dt} = kP$$

Solving:

$$\frac{dP}{P} = kdt \qquad \text{(Separate the variables)}$$

$$\int \frac{1}{P}\,dP = \int kdt \qquad \text{(Integrate both sides)}$$

$$\ln|P| = kt + C \qquad \text{(Apply integration formulas)}$$

$$e^{\ln|P|} = e^{kt+C} \qquad \text{(Raise both sides to exponential function of base } e\text{)}$$

$$|P| = e^{kt}e^{C} \qquad \text{(Use inverse property } e^{\ln k} = k \text{ and law of exponents } b^{x+y} = b^x b^y\text{)}$$

$$P(t) = Ae^{kt} \qquad \text{(Use absolute value definition } P = \pm e^{C}e^{kt} \text{ and replace constant } \pm e^{C} \text{ with } A.\text{)}$$

Solving:

The equation $P(t) = Ae^{kt}$ represents the general solution of the differential equation. Using the initial condition $P(0) = P_0$, we can find the particular solution.

$P_0 = P(0) = Ae^{k(0)}$ (Substitute $t = 0$ in the equation and equate to P_0)

$P_0 = A(1)$ (Note that $e^{k(0)} = e^0 = 1$)

$A = P_0$ (Solve for A)

Hence, $P(t) = P_0 e^{kt}$ is the particular solution.

Example 4:

The population of a community is known to increase at a rate proportional to the number of people present at a time t. If the population has doubled in 6 years, how long it will take to triple?

Solution :

Let N(t) denote the population at time t. Let N(0) denote the initial population (population at t=0).

$$\frac{dN}{dt} = kN(t)$$

$N(t)=Ae^{kt}$, where $A=N(0)$

$Ae^{6k}=N(6) = 2N(0) = 2A$

or $e^{6k}=2$ or $k = 6 \ln 2$

Find t when $N(t)=3A=3N(0)$

or $N(0)\, e^{kt}=3N(0)$

Example 4:

$$3 = e^{\frac{1}{6}(\ln 2)t}$$

$$\ln 3 = \frac{(\ln 2)t}{6}$$

$$t = \frac{6 \ln 3}{\ln 2}$$

$\simeq$9.6 years (approximately 9 years 6 months)