Lecture 8 : TAYLOR POLYNOMIALS

Numerical analysis uses results and methods from many areas of mathematics, particularly those of calculus and linear algebra. Taylor's theorem is very useful tool from calculus. Taylor polynomials are used to evaluate other functions approximately and Taylor theorem is used for the estimation of error in these polynomial approximations.

The Taylor polynomial

Most functions f(x) that occur in mathematics, like $f(x) = \cos x$, e^x, or $\sqrt{x}$, cannot be evaluated exactly in any simple way. In such cases, functions f(x) are used that are almost equal to f(x) and are easier to evaluate. They are easy to work with and they are usually an efficient means of approximating f(x). A related form of function is the piecewise polynomial function and it also is widely used in applications.

Among polynomials, the most widely used is the Taylor polynomial. Taylor polynomial is easy to construct and it is often a first step in obtaining more efficient approximations.

Let f(x) be a given function. The Taylor polynomial is constructed to mimic the behavior of f(x) at some point x = a. As a result, it will be nearly equal to f(x) at points x near a.

Linear Taylor polynomial :
To find a linear polynomial $p_1(x)$ for a function f(x), first we must treat $p_1(a) = f(a)$, and $p'_1(x) = f'(a)$.

Then polynomial $p_1(x)$ is given by

$$\mathbf{p_1(x) = f(a) + (x - a)f'(a)} \qquad \ldots\ldots\ (1)$$

The graph of $y = p_1(x)$ is tangent to that of $y = f(x)$ at $x = a$.

Example : Find the polynomial for the function $f(x) = e^x$ and $a = 0$.
Then $f(a) = e^0 = 1$, and $f'(a) = e^0 = 1$.

$$p_1(x) = f(a) + (x - a)f'(a) = 1 + (x - 0).1 = 1 + x.$$

If we draw the graphs for the two functions f(x) and $p_1(x)$, the graph $p_1(x)$ is a straight line acting as a tangent to the curve $f(x) = e^x$.

Quadratic Taylor polynomial : To find a quadratic polynomial $p_2(x)$ that approximates $f(x)$ near $x = a$:
Since there are three coefficients in the formula of a quadratic polynomial, we write the polynomial $p_2(x)$ as, $p_2(x) = b_0 + b_1 x + b_2 x^2$.

To mimic the behavior of f(x) at x = a, we define

$$p_2(x) = f(a); \quad p'_2(x) = f'(a); \quad p''_2(x) = f''(a).$$

Then the corresponding formula for the polynomial $p_2(x)$ will be

$$p_2(x) = f(a) + (x-a)f'(a) + (1/2)(x-a)^2 f''(a).$$

If the graphs of f(x), $p_1(x)$ and $p_2(x)$ are checked, the graph of $p_2(x)$ will mimic the behavior of f(x) more than $p_1(x)$ [which is simply a tangent to f(x)].

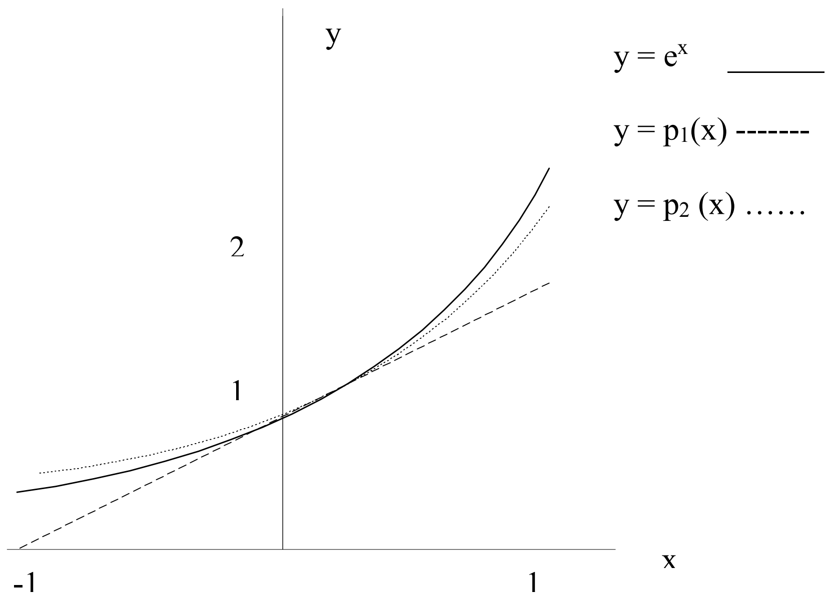

Fig. 1. Linear and quadratic Taylor approximations

$p_n(x)$, the polynomial of degree n :

If we continue the process of mimicking the behavior of f(x) at x = a, we get a polynomial $p_n(x)$ of degree n, called the *Taylor polynomial of degree n* for the function f)x_ and the point of approximation a, as

$$p_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a)$$

$$= \sum_{j=0}^{n} \frac{(x-a)^j}{j!} f^{(j)}(a) \qquad \ldots\ldots \quad (2)$$

where $p_n^{(j)}(a) = f^{(j)}(a)$, for $j = 0, 1, \ldots, n$.

In the formula, $f^{(0)}(a) = f(a)$, and

$$j! = \begin{cases} 1, & j = 0 \\ j(j-1)\ldots\ldots(2)(1), & j = 1,2,3,4,\ldots.. \end{cases}$$

For example, let $f(x) = e^x$ and $a = 0$. Then

$$f^{(j)}(x) = e^x, \quad f^{(j)}(0) = 1, \quad \text{for all } j \geq 0$$

Thus,

$$p_n(x) = 1 + x + \frac{1}{2!}x^2 + \ldots. + \frac{1}{n!}x^n$$

$$= \sum_{j=0}^{n} \frac{x^j}{j!}$$

The error in Taylor's polynomial

The approximation error in Taylor polynomial, $R_n(x)$ is equal to the difference between the original function $f(x)$ and the Taylor polynomial $p_n(x)$. The error (or remainder) is given by the formula,

$$\mathbf{f(x) - p_n(x) = R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c_x), \quad \alpha \leq x \leq \beta} \qquad \ldots.(3)$$

assuming that $f(x)$ has $n+1$ continuous derivatives on an interval $\alpha \leq x \leq \beta$, and let the point a belong to that interval. c_x is an unknown point between a and x.

For example, for the function, $f(x) = e^x$ and $a = 0$ for which the Taylor polynomial is given in equation (2), the approximation error is given by

$$e^x - p_n(x) = \frac{x^{n+1}}{(n+1)!} e^c, \qquad n \geq 0 \qquad \ldots\ldots.(4)$$

With c between 0 and x. From this formula, we can prove that for each fixed x, the error tends to 0 as $n \rightarrow \infty$; this should be intuitively clear when $|x| \leq 1$. Also from the formula, it appears that for each fixed value of n, the error becomes larger as x moves away from 0.

As a special case of equl (4), let $x = 1$. Then from equation (3),

$$e \sim p_n(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots. + \frac{1}{n!}$$

Then from equation (3),

$$e^x - p_n(1) = R_n(1) = \frac{x^{n+1}e^c}{(n+1)!}, \qquad 0 < c < 1 \qquad \ldots\ldots.(4)$$

Since e < 3, we can bound $R_n(1)$ as follows:

$$\frac{x^{n+1}}{(n+1)!} \leq R_n(1) \leq \frac{e}{(n+1)!} < \frac{3}{(n+1)!}$$

Polynomial evaluation

The evaluation of a polynomial would appear to be a straightforward task. From a programmer's perspective, (i) *the simplest method of evaluation* is to compute each term independently of the remaining terms. For example, the term cx^k is computed in a program by

c * x ^ k, or c * x ** k

depending on which computer language is being used. This requires k multiplications with most compilers.

For example, for the evaluation of $p(x) = 3 - 4x - 5x^2 - 6x^3 + 7x^4 - 8x^5$, with this approach, there will be $1 + 2 + 3 + 4 + 5 = 15$ multiplications.

There is a *(ii)* second method of evaluation, wherein each power of x is computed by multiplying x with the preceding power of x. For example, $x^3 = x(x^2)$, $x^4 = x(x^3)$, $x^5 = x(x^4)$. Each term cx^k takes two multiplications for k > 1.

Then, the evaluation of p(x) uses $1 + 2 + 2 + 2 + 2 = 9$ multiplications, which is considerable savings over the first method, especially with higher-degree polynomials.

There is a (iii) third method of evaluation, called *nested multiplication*, with which the polynomial p(x) can be written and evaluated in the form,

$$P(x) = 3 + x(-4 + x(-5 + x(-6 + x(7 - 8x)))).$$

In this third method, the number of multiplications is only 5, an additional saving over the second method of evaluation. The nested multiplication method is the preferred evaluation procedure; and its advantage increases as the degree of the polynomial becomes larger.

Consider the general polynomial of degree n,

$$p(x) = a_0 + a_1x + \ldots. + a_nx^n, \qquad a \text{ not equal to } 0.$$

If we use the second method, the number of multiplications in the evaluation of p(x) equals 2n – 1. In the case of the nested multiplication method, p(x) can be written as

$$p(x) = a_0 + x(a_1 + x(a_2 + \ldots + x(a_{n-1} + a_n x) \ldots))$$

This uses only n multiplications, a savings of about 50% over the second method.